B3.2 GEOMETRY OF SURFACES - WEEK 0 (NOT TO HAND IN)

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Remark. Reading lecture notes is surprisingly helpful.

Have a look at the introductory Chapter 1 of the lecture notes, which contains many pictures of surfaces. You may also find helpful the *Analysis and Topology Dictionary – Handout*, which recalls some useful terminology from Prelims and Part A.

Exercise 1. Topological surfaces by gluing the sides of a square.

Starting from the square, how many different topological spaces can you produce, if you are allowed to identify the sides? (make as many identifications as you like, in pairs or more)

Which of these topological spaces do you think are topological surfaces?

You may have heard that it is enough to specify a direction along which you identify the edges. In what precise sense is this true, and why?

Exercise 2. The real projective plane

Recall that $\mathbb{RP}^2$ can be thought of as the collection of all 1-dimensional vector subspaces inside $\mathbb{R}^3$. Explain why a 2-dimensional vector subspace of $\mathbb{R}^3$ determines a curve in $\mathbb{RP}^2$. This is called a *line* in $\mathbb{RP}^2$.

Show that two distinct lines in $\mathbb{RP}^2$ always intersect in exactly one point.

Show that through two points there passes a unique line.

By viewing $\mathbb{RP}^2$ as the 2-sphere modulo the action of the antipodal map $x \mapsto -x$, show that every line in $\mathbb{RP}^2$ is homeomorphic to a circle. Convince yourself that, using the notion of distance on $\mathbb{RP}^2$ inherited from S^2, the shortest path connecting two points of $\mathbb{RP}^2$ is along the unique line through those two points.

Exercise 3. The inverse and implicit function theorems

Read Chapter 7 of the lecture notes, about the inverse and implicit function theorems (you may also find the lecture notes of the short option *Introduction to Manifolds* helpful).

Let $A(t)$ be an $n \times n$ matrix that depends smoothly on $t \in \mathbb{R}$. Suppose λ_0 is an eigenvalue of $A(0)$ with[1] algebraic multiplicity 1. Show that, for small t, $A(t)$ has a smoothly varying eigenvalue $\lambda(t)$ such that $\lambda(0) = \lambda_0$.

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[1] *Challenge Question.* What can you say if the multiplicity is not 1? It may help to play with Matlab/Mathematica, by picking a path of matrices and then making it plot the eigenvalues, so you see the curves they sweep in the complex plane. An interesting case is when $A(t)$ is holomorphic in $t \in \mathbb{C}$ (e.g. polynomial entries in t).