B3.2 GEOMETRY OF SURFACES - EXERCISE SHEET 1

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Exercise 1. $\mathbb{CP}^1$ as a quotient of spheres.

Recall that the complex projective space $\mathbb{C}P^1$ is the space of complex lines through 0 in $\mathbb{C}^2$. By thinking about how a complex 1-dimensional vector space intersects the sphere $S^3 \subset \mathbb{R}^4 = \mathbb{C}^2$, show that $\mathbb{C}P^1$ as a topological surface can be viewed as a quotient

$$\mathbb{C}P^1 = S^3/S^1$$

where you need to explain how the group S^1 acts on S^3.

Exercise 2. The Möbius band.

The open Möbius band is the quotient

$$M = [0,1] \times (0,1) \,/\, ((0,y) \sim (1,1-y) \text{ for all } y \in (0,1)).$$

Briefly explain why M is a smooth surface. Find (a homeomorphic copy of) M inside the real projective space $\mathbb{R}P^2$ and inside the Klein bottle K. The Möbius band $\overline{M}$, is obtained by replacing $(0,1)$ by $[0,1]$ above. Show[1] that the boundary of $\overline{M}$ is homeomorphic to S^1. Show that $\mathbb{R}P^2 =$ (closed disc) $\cup\, \overline{M}$ glued along the circular boundary, and $K = \overline{M} \cup \overline{M}$.

Exercise 3. Riemann surfaces arising from polynomial equations.

Briefly explain a natural way to make the sets

(1) $S_1 = \{(z,w) \in \mathbb{C}^2 : w^2 = (z-1)(z-2)\} \cup \{+\infty\} \cup \{-\infty\}$
(2) $S_2 = \{(z,w) \in \mathbb{C}^2 : w^2 = (z-1)(z-2)(z-3)\} \cup \{\infty\}$

into Riemann surfaces. Find homeomorphisms $S_1 \cong$ sphere, $S_2 \cong$ torus. *(Hints in footnote*[2])

Exercise 4. The Klein bottle as a quotient of $\mathbb{R}^2$.

Consider the quotient

$$S = \mathbb{R}^2/G$$

where $G = \mathbb{Z}^2$ acts[3] by $(n,m) \bullet (x,y) = ((-1)^m x + n, y + m)$ on $\mathbb{R}^2$, where $n, m \in \mathbb{Z}$. Briefly explain why S is a smooth surface. Show that S is homeomorphic to the Klein bottle.

P.T.O.

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[1] *Hint. Try cutting out a clever disc from* $\mathbb{R}P^2$.

[2] Hints. It helps if you first ask yourself what local holomorphic coordinate you would use at solutions (z,w) of $w^2 = z$ (recall from lecture notes the discussion of the square root $z^{1/2}$). Then try to build the solution set S_1 by gluing two cut-domains: two copies of $\mathbb{C}$ cut from 1 to 2. Just like for $\operatorname{Log} z$ in lectures, each subset you cut gives rise to *two* copies of that subset in the Riemann surface. In order to be able to draw the Riemann surface inside $\mathbb{R}^3$, it is convenient to reflect one of the cut-domains about the x-axis. Near infinity, try using the coordinates $X = \frac{1}{z}$ and $Y = \frac{w}{z}$ instead of z, w, and ask yourself what happens for $X = 0$ (corresponding to "$z = \infty$"). For S_2 you will need a second cut, from 3 to ∞, and try instead $Y = \frac{w}{z^2}$.

[3] $G = \mathbb{Z}^2$ as a set, but as a group $G = \mathbb{Z} \rtimes_\varphi \mathbb{Z}$ is a *semi-direct product* where $\varphi : \mathbb{Z} \to \operatorname{Aut}(\mathbb{Z})$, $\varphi(m) = (-1)^m$. Try checking the conditions of a "group action", and you will see what group operation you must use on the set $\mathbb{Z}^2$. In any case, $S = \mathbb{R}^2/\sim$ for the equivalence relation $(x,y) \sim ((-1)^m x + n, y + m)$ for all $n, m \in \mathbb{Z}$.

Exercise 5. The space of lines in $\mathbb{R}^2$.

Let S be the set of all straight lines in $\mathbb{R}^2$ (not necessarily through 0). Show that there is a natural[4] way to make S into a topological surface. Show that S is homeomorphic to the open Möbius band M.

Exercise 6. Tori and figure 8 loops.

A figure 8 loop consists of two circles touching at a point. Show that a torus can be obtained by attaching a disc onto a figure 8 loop.

Exercise 7. The Euler characteristic constrains graphs.

Given five points in the plane, show that it is impossible to connect each pair by paths which do not cross. Is it possible for five points in a torus?

Exercise 8. The Euler characteristic constrains Platonic solids.

Using the Euler characteristic, show that there are no more than five Platonic solids.[5]

Exercise 9. Bump functions and embedding the Klein bottle in $\mathbb{R}^4$.

Recall from analysis that $\alpha(x) = e^{-1/x^2}$ *is a function* $\mathbb{R} \to \mathbb{R}$ *(defining it to be zero at* $x = 0$*) which is infinitely differentiable, so smooth, but all the derivatives at* $x = 0$ *vanish! (so the Taylor series at* 0 *is useless)*

⋄ Sketch the following functions (no need to justify your sketches):

(1) $\beta(x) = \alpha(x)$ for $x > 0$, and $\beta(x) = 0$ for $x \leq 0$.
(2) $\gamma(x) = \beta(x - a) \cdot \beta(b - x)$, where $0 < a < b$.
(3) $\delta(x) = \int_x^b \gamma(t)\,dt / \int_a^b \gamma(t)\,dt$.

⋄ The function $\varepsilon : \mathbb{R}^n \to \mathbb{R}$, $\varepsilon(x) = \delta(\|x\|)$ is called a *bump function compactly supported on the disc* $\|x\| \leq b$. Check that $\varepsilon(\mathbb{R}^n) \subset [0, 1]$, that $\varepsilon = 1$ on $\|x\| \leq a$ and $\varepsilon = 0$ on $\|x\| \geq b$.

⋄ A figure 8 loop can be obtained as the image of a continuous map $f : S^1 \to \mathbb{R}^2$ which is injective except at $\pm 1 \in S^1 \subset \mathbb{C}$ where $f(+1) = f(-1)$ (so f is not an embedding). Using a bump function, show that the figure 8 loop can be continuously embedded into $\mathbb{R}^3$.

⋄ Show that the Klein bottle K can be smoothly embedded in $\mathbb{R}^4$.

⋄ **Optional harder question:** Show that all compact surfaces can be embedded in $\mathbb{R}^4$.

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[4]*Hint. For example, lines which are not vertical can be parametrized by 2 numbers: the angle* $\theta \in (-\pi/2, \pi/2)$ *which tells you how much the line is tilted, and* $r \in \mathbb{R}$ *which is the signed distance of the line from the origin* $0 \in \mathbb{R}^2$ *(using the* $+$ *sign if the line passes above* 0*, and the* $-$ *sign if it passes below* 0*).*

[5]A Platonic solid is a convex polyhedron with congruent faces consisting of regular polygons and the same number of faces meet at each vertex.