B3.2 GEOMETRY OF SURFACES - EXERCISE SHEET 4

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Exercise 1. Holomorphic maps between Riemann surfaces.

Using the local form of a holomorphic map between Riemann surfaces, deduce:
Open mapping theorem: any holomorphic map $f : R \to S$ between Riemann surfaces, with R connected, is either constant or an open map, meaning f(any open set) is open.[1]

Deduce the following, for $f : R \to S$ holomorphic, R, S Riemann surfaces:

(1) If f is non-constant, R compact connected, then $f(R) \subset S$ is a connected component.
(2) If f is non-constant, R, S both compact connected, then f is surjective: $f(R) = S$.
(3) If R is compact connected, S non-compact connected, then f is constant.
(4) A holomorphic map $S \to \mathbb{C}$ on a compact connected Riemann surface is constant.
(5) Fundamental theorem of algebra: non-constant complex polynomials have a root.

Exercise 2. Riemann-Hurwitz formula.
In the following, all spaces are compact connected Riemann surfaces, and all maps are holomorphic maps. Deduce from the Riemann-Hurwitz formula that:

(1) if $f : R \to S$ is not constant, then the genus $g(R) \geq g(S)$.
(2) if $f : \mathbb{C}P^1 \to S$ is not constant, then S is homeomorphic to a sphere.
(3) if $f : R \to S$ has degree 1 then f is a biholomorphism.
(4) if R admits a meromorphic function with only one pole of order 1, then $R \cong \mathbb{C}P^1$.

Exercise 3. Implicit function theorem.

Consider $R = \{(z, w) \in \mathbb{C}^2 : w^3 = z^3 - z\}$. Use the implicit function theorem to check that R is a Riemann surface. Now consider the projection $\pi : R \to \mathbb{C}$, $\pi(z, w) = z$. Find the branch points of π and the valency $v_\pi(p)$ at the ramification points.

Next, we seek how many points are "missing" at infinity. Write $z^3 - z = z^3(1 - z^{-2})$ for large $|z|$, and briefly explain that there are three holomorphic solution functions to $w^3 = z^3 - z$. Deduce that $\pi^{-1}(\{z \in \mathbb{C} : |z| > 100\})$ is biholomorphic to three punctured discs.

Compute the Euler characteristic of R using the Riemann-Hurwitz formula. Deduce that R is homeomorphic to a torus with three points removed.

Exercise 4. Meromorphic functions on Riemann surfaces.

Show that a map $f : S \to \mathbb{C}P^1$ is meromorphic if and only if locally f is expressible as a quotient of holomorphic functions (where the denominator is not identically zero).

Show that if f, g are two meromorphic functions on a compact connected Riemann surface having the same zeros and the same poles (including multiplicities) then $f = \text{constant} \cdot g$.

By comparing Taylor series of $\wp, \wp'$ near ramification points, deduce by the previous part (by viewing the two sides of the equation below as meromorphic functions) that:

$$\wp'(z)^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3)$$

where $e_1 = \wp(\frac{1}{2}\omega_1)$, $e_2 = \wp(\frac{1}{2}\omega_2)$, $e_3 = \wp(\frac{1}{2}(\omega_1 + \omega_2))$, $\infty = \wp(0)$ are the branch points of $\wp$.

Please turn over.

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[1] *Hint. Notice that to show a map is open, it's enough to show that for each p, there are some nice arbitrarily small open neighbourhoods of p which map to open sets.*

Exercise 5. Elliptic curves and the Weierstrass $\wp$-function.

The goal is to prove that the following is a biholomorphism:

$$\begin{array}{rcl} \mathbb{C}/\Lambda & \to & S = \{(Z,W) \in \mathbb{C}^2 : W^2 = 4(Z-e_1)(Z-e_2)(Z-e_3)\} \cup \{\infty\} \\ z & \mapsto & (\wp(z), \wp'(z)) \end{array}$$

where on the right we compactify as done in Exercise Sheet 1. *Here is a checklist/hints:*

(1) Explain why e_1, e_2, e_3 are distinct,
(2) Show S is a Riemann surface. In particular, what is the local holomorphic coordinate?
(3) Explain why the map is well-defined,
(4) Show that the map is holomorphic (do this carefully, locally),
(5) For very general reasons, explain why the map has to be surjective,
(6) Show that the degree of the map is 1, and use Exercise 2.

Exercise 6. Hyperbolic Geometry.

For $k \in (0, \infty) \subset \mathbb{R}$, show that the dilation $\mathbb{H} \to \mathbb{H}$, $z \mapsto kz$ is an isometry, by directly verifying that the hyperbolic metric is preserved.

Verify directly that the geodesic equation holds for the curve $\gamma : \mathbb{R} \to \mathbb{H}$, $t \mapsto e^t i$.

Find the locus of all points in $\mathbb{H}$ that are equidistant from γ (by a given fixed distance).

Describe the locus of all points in $\mathbb{H}$ equidistant from a general geodesic in $\mathbb{H}$. *(You may use your knowledge of the isometries of* $\mathbb{H}$ *and the geodesics in* $\mathbb{H}$*.)*