B3.1 Galois Theory Sheet 0 (MT 2018)

Exercises marked with * are slightly more difficult. Polynomials.

1. Revise Eisenstein criterion. Give an example (with proof) of a polynomial which is irreducible over $\mathbb{Q}$ but does not satisfy Eisenstein criterion.

2. Let $\mathbb{Q}$ be the field of rational numbers. Show that $x^4 + x^3 + x^2 + x + 1 \in \mathbb{Q}[x]$ is an irreducible polynomial.

3. Show that $x^{888} - 999 \in \mathbb{Q}[x]$ is irreducible.

4. Show that $1 - 999x^{888} \in \mathbb{Q}[x]$ is irreducible.

5. Show that $x^8 + 1$ splits into linear factors over the field $\mathbb{Z}/17\mathbb{Z}$.

6. Let $F = \mathbb{Z}/17\mathbb{Z}$. Show that $F[x]/(x^8+1)$ and F^8 are isomorphic as rings (multiplication on F^8 being componentwise).

Group Theory. Revise/study the notion of group action on a set and the notion of normal subgroup. Let G be a finite group acting on a finite set X.

(a) Let $x \in X$, show that the stabilizer of x

$$S_x = \{g \in G : \ gx = x\}$$

is a subgroup of G.

(b) *Orbit-stabiliser theorem.* For any $x \in X$, let $\mathcal{O}(x) = \{gx : \ g \in G\} \subseteq X$ be the orbit of x under the action given by G. Show that $\mathcal{O}(x) = |G|/|S_x|$ by showing that there is a bijection between the cosets of S_x and the elements of $\mathcal{O}(x)$.

(c) Show that if $\mathcal{O}(x) \neq \mathcal{O}(y)$, then $\mathcal{O}(x) \cap \mathcal{O}(y) = \emptyset$ and therefore that there exists a subset Y of X such that $X = \bigsqcup_{y \in Y} \mathcal{O}(y)$.

(d*) Suppose that G acts transitively on X (i.e. for any $x \in X$, $\mathcal{O}(x) = X$). In addition, suppose that $|X| > 1$. Show that there exists $g \in G$ such that $gx \neq x$ for any $x \in X$.

(e) Let $g \in G$. Define a map $\psi_g : G \to G$ as follows: for any $h \in G$, $\psi_g(h) = ghg^{-1}$. Show that ψ_g is an automorphism and that $g \mapsto \psi_g$ is an homomorphism of G in $\mathrm{Aut}(G)$. Let $H = \{\psi_g \,|\, g \in G\}$ (one usually refers to H as the group of *inner automorphisms* of G). Show that H is a group and that is normal in $\mathrm{Aut}(G)$.

(f*) Let $Z(G) = \{g \in G : ghg^{-1} = h \ \forall h \in G\}$ be the center of G. Show that $Z(G)$ is normal. In addition, show that if $G/Z(G)$ is cyclic, then G is abelian.

(g*) Using (e) and (f) (or otherwise) show that if $\mathrm{Aut}(G)$ is cyclic then G is abelian.