B3.1 Galois Theory Sheet 2 (MT 2018)

In these problems K denotes an arbitrary field, $K[x]$ denotes the ring of polynomials in one variable x over K and $K(x)$ the ring of rational functions in the variable x (i.e. the fraction field of $K[x]$). If p is a prime number, then $\mathbb{F}_p$ denotes the field of integers modulo p. Recall the multiplicative group of $\mathbb{F}_p$ is cyclic.

1. Let K be a finite field. Show that there exists a positive integer d and a prime number p such that $|K| = p^d$. Hint: what is the prime subfield of K?

2. Factorise $f(x) = x^6+x^3+1$ into irreducible factors over K for each of $K = \mathbb{F}_2, \mathbb{F}_3, \mathbb{F}_{19}, \mathbb{Q}$.

   Calculate the formal derivative Df. Over which of these fields K do the irreducible factors of f have distinct roots in any splitting field for f?

3. Show that if f is a polynomial of degree n over K, then its splitting field has degree less than or equal to $n!$ over K.

4. Find the degrees of the splitting fields of the following polynomials.

   (a) $x^3 - 1$ over $\mathbb{Q}$;

   (b) $x^3 - 2$ over $\mathbb{Q}$;

   (c) $x^5 - t$ over $\mathbb{F}_{11}(t)$;

5. Let $L = \mathbb{Q}(2^{1/3}, 3^{1/4})$. Compute the degree of L over $\mathbb{Q}$.

6. Recall that $\alpha \in \mathbb{C}$ is algebraic over $\mathbb{Q}$ if α satisfies a (monic) polynomial over $\mathbb{Q}$, equivalently if $[\mathbb{Q}(\alpha) : \mathbb{Q}] < \infty$. Let

$$\mathbb{A} = \{\alpha \in \mathbb{C} : \alpha \text{ is algebraic over } \mathbb{Q}\}.$$

   (a) Show that $\mathbb{A}$ is the union of all the subfields L of $\mathbb{C}$ which are finite extensions of $\mathbb{Q}$.

   (b) Prove that $\mathbb{A}$ is a subfield of $\mathbb{C}$. [Hint: if $\alpha, \beta \in \mathbb{A}$, consider the extension $\mathbb{Q}(\alpha, \beta) : \mathbb{Q}$.]

   (c) Prove that $\mathbb{A} : \mathbb{Q}$ is not a finite extension.

7. Which of the following fields are normal extensions of $\mathbb{Q}$?

   (a) $\mathbb{Q}(\sqrt{2}, \sqrt{3})$;

   (b) $\mathbb{Q}(2^{1/4})$;

   (c) $\mathbb{Q}(\alpha)$, where $\alpha^4 - 10\alpha^2 + 1 = 0$.