B2.1 Introduction to Representation Theory
Problem Sheet 1, MT 2018

1. Let G be a finite group and k be a field. Let kG be the group algebra as defined in the notes. Let $\mathcal{F}(G,k)$ be the k-vector space of functions $f : G \to k$. Endow $\mathcal{F}(G,k)$ with a ring structure given by the *convolution*:

$$(f_1 \star f_2)(g) = \sum_{h \in G} f_1(gh^{-1}) f_2(h), \quad f_1, f_2 \in \mathcal{F}(G,k).$$

Prove that $\mathcal{F}(G,k)$ and kG are isomorphic as k-algebras. (N.B.: When considered with the pointwise multiplication, $\mathcal{F}(G,k)$ is *not* isomorphic to kG.)

2. Let A be a ring. An element $e \in A$ is called an *idempotent* if $e^2 = e$. Verify that eAe is a subring of A and that

$$\mathrm{End}_A(Ae) \cong eA^{\mathrm{op}}e \text{ (as rings)},$$

where $Ae = \{ae \mid a \in A\}$ is the left A-module under left multiplication by A.

3. Let G be a finite group and $\rho : G \to GL(V)$ be a G-representation on the k-vector space V. Recall that a *G-stable* subspace of V is a vector subspace $U \subset V$, such that $\rho(g)(U) \subseteq U$ for all $g \in G$.

Let S_n be the symmetric group of permutations in n letters. Consider the natural permutation representation of S_n on $V = \mathbb{C}^n$, $\rho : S_n \to GL(\mathbb{C}^n)$,

$$\rho(\sigma)(x_1, x_2, \ldots, x_n) = (x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}),\ x_1, x_2, \ldots, x_n \in \mathbb{C}.$$

Determine all the S_n-stable subspaces of V.

4. Let A be an algebra over a field k with identity 1_A. Recall that a subspace B of A is called a *subalgebra* if $1_A \in B$, and whenever $b_1, b_2 \in B$, this implies that $b_1 \cdot b_2 \in B$. The *centre* of A is defined to be the set

$$Z(A) = \{x \in A \mid ax = xa \text{ for all } a \in A\}.$$

   (a) Show that $Z(A)$ is a subalgebra of A.
   (b) Let $A = A_1 \times A_2$ be the product of algebras A_i, $i = 1, 2$. Identify the centre $Z(A)$ in terms of the centres $Z(A_1)$ and $Z(A_2)$.
   (c) Show that the centre of $M_n(k)$ consists precisely of the scalar multiples of the identity matrix.

5. Let G be a finite group. We determine a basis for the centre of the group algebra $\mathbb{C}G$. Assume that G has s conjugacy classes, denoted by $\mathcal{C}_1, \ldots, \mathcal{C}_s$. Define the elements $C_i = \sum_{x \in \mathcal{C}_i} x$ in the group algebra $\mathbb{C}G$.

   (a) Show that $C_i \in Z(\mathbb{C}G)$.

(b) Show that $\{C_1, \ldots, C_s\}$ is a basis of $Z(\mathbb{C}G)$.

6. Suppose A is a k-algebra and V is some A-module, let $\theta : A \to \mathrm{End}_k(V)$ be the corresponding representation. Assume that U is a submodule of V. Show that there is a basis of V such that for every $a \in A$ the matrix of $\theta(a)$ has block form

$$\theta(a) = \begin{pmatrix} \theta_1(a) & \theta_2(a) \\ 0 & \theta_3(a) \end{pmatrix}$$

where θ_1 and θ_3 describe the actions on U and on V/U. Suppose there is such basis for which $\theta_2(a) = 0$ for all $a \in A$. Show that then V is the direct sum $V = U \oplus W$ where W is some submodule of V.

7. Let k be a field of prime characteristic p, let G be a finite group and Ω a G-set. We assume that G acts transitively on Ω, that is, for any $x, y \in \Omega$, there exists $g \in G$ such that $gx = y$. We consider the following two subsets of the permutation module $M = k\Omega$:

$$\begin{aligned} M_1 &:= k \cdot (\sum_{\omega \in \Omega} b_\omega), \\ M_2 &:= \left\{ \sum_{\omega \in \Omega} \lambda_\omega b_\omega \in M \mid \sum_{\omega \in \Omega} \lambda_\omega = 0 \right\}. \end{aligned}$$

(a) Show that M_1 and M_2 are submodules of M. What are the vector space dimensions of M_1 and M_2? Describe the representations corresponding to M_1 and M/M_2 respectively.

(b) Prove that M_1 is a direct summand of M if and only if p is coprime to $|\Omega|$.

(c) Assume that $\mathrm{char}(k) = p$ is a divisor of the order of G and let $\Omega = G$. Prove that the trivial module M_1 is a submodule of the regular module kG. Show that M_1 has no complement in kG, that is, there exists no submodule T of kG with $kG = M_1 \oplus T$.