

13. The Completeness Theorem for Predicate Calculus

13.1 Theorem (Gödel)

Let $\Gamma \subseteq \text{Form}(\mathcal{L})$, $\phi \in \text{Form}(\mathcal{L})$.

If $\Gamma \models \phi$ then $\Gamma \vdash \phi$.

Two additional assumptions:

- Assume all $\gamma \in \Gamma$ and ϕ are *sentences* – the Theorem is true more general, but the proof is much harder and applications are typically to sentences.
- Further assumption (for the start – later we do the general case): *no $\doteq$ -symbol in any formula of Γ or in ϕ .*

First Step

Call $\Delta \subseteq \text{Sent}(\mathcal{L})$ **consistent** if for no sentence ψ , both $\Delta \vdash \psi$ and $\Delta \vdash \neg\psi$.

13.2 enough to show

($\star$) *Every consistent set of sentences has a model.*

i.e. Δ consistent $\Rightarrow$
there is an $\mathcal{L}$ -structure $\mathcal{A}$ such that
 $\mathcal{A} \models \delta$ for every $\delta \in \Delta$.

Proof: Assume $\Gamma \models \phi$ and assume ($\star$)
 $\Rightarrow \Gamma \cup \{\neg\phi\}$ has no model
 $\Rightarrow_{(\star)} \Gamma \cup \{\neg\phi\}$ is not consistent
 $\Rightarrow \Gamma \cup \{\neg\phi\} \vdash \psi$ and $\Gamma \cup \{\neg\phi\} \vdash \neg\psi$ for some ψ
 $\Rightarrow_{\text{DT}} \Gamma \vdash (\neg\phi \rightarrow \psi)$ and $\Gamma \vdash (\neg\phi \rightarrow \neg\psi)$ for some ψ
But $\Gamma \vdash ((\neg\phi \rightarrow \psi) \rightarrow ((\neg\phi \rightarrow \neg\psi) \rightarrow \phi))$ [taut.]
 $\Rightarrow \Gamma \vdash \phi$ [2xMP] $\square_{13.2}$

Second Step

We shall need an *infinite* supply of constant symbols.

To do this, let ϕ' be the formula obtained by replacing every occurrence of c_n by c_{2n} .

For $\Delta \subseteq \text{Form}(\mathcal{L})$ let

$$\Delta' := \{\phi' \mid \phi \in \Delta\}$$

Then

13.3 Lemma

- (a) Δ consistent $\Rightarrow \Delta'$ consistent
- (b) Δ' has a model $\Rightarrow \Delta$ has a model.

Proof: easy exercise.

Third Step

- $\Delta \subseteq \text{Sent}(\mathcal{L})$ is called **maximal consistent** if Δ is consistent, and for any $\psi \in \text{Sent}(\mathcal{L})$: $\Delta \vdash \psi$ or $\Delta \vdash \neg\psi$.
- $\Delta \subseteq \text{Sent}(\mathcal{L})$ is called **witnessing** if for all $\psi \in \text{Form}(\mathcal{L})$ with $\text{Free}(\psi) \subseteq \{x_i\}$ and with $\Delta \vdash \exists x_i \psi$ there is some $c_j \in \text{Const}(\mathcal{L})$ such that $\Delta \vdash \psi[c_j/x_i]$

to prove CT:

13.4 enough to show:

Every maximal consistent witnessing set of sentences has a model.

For the proof of 13.4 we need 2 Lemmas:

13.5 Lemma

If $\Delta \subseteq \text{Sent}(\mathcal{L})$ is consistent, then for any sentence ψ , either $\Delta \cup \{\psi\}$ or $\Delta \cup \{\neg\psi\}$ is consistent.

Proof: Exercise – as for Propositional Calculus. $\square$.

13.6 Lemma

Assume $\Delta \subseteq \text{Sent}(\mathcal{L})$ is consistent, $\exists x_i \psi \in \text{Sent}(\mathcal{L})$, $\Delta \vdash \exists x_i \psi$, and c_j is not occurring in ψ nor in any $\delta \in \Delta$.

Then $\Delta \cup \{\psi[c_j/x_i]\}$ is consistent.

Proof:

Assume, for a contradiction, that there is some $\chi \in \text{Sent}(\mathcal{L})$ such that

$$\Delta \cup \{\psi[c_j/x_i]\} \vdash \chi \text{ and } \Delta \cup \{\psi[c_j/x_i]\} \vdash \neg\chi.$$

May assume that c_j does *not* occur in χ
(since $\vdash (\chi \rightarrow (\neg\chi \rightarrow \theta))$ for *any* sentence θ).

By DT, $\Delta \vdash (\psi[c_j/x_i] \rightarrow \chi)$
and $\Delta \vdash (\psi[c_j/x_i] \rightarrow \neg\chi)$.

Then also

$$\Delta \vdash (\psi \rightarrow \chi) \text{ and } \Delta \vdash (\psi \rightarrow \neg\chi)$$

(Exercise Sheet # 4 (2)(ii))

By $\forall$, $\Delta \vdash \forall x_i(\psi \rightarrow \chi)$
and $\Delta \vdash \forall x_i(\psi \rightarrow \neg\chi)$
(note that $x_i \notin \text{Free}(\delta)$ for any $\delta \in \Delta \subseteq \text{Sent}(\mathcal{L})$).

Now: $\vdash (\forall x_i(A \rightarrow B) \rightarrow (\exists x_i A \rightarrow B))$
for any $A, B \in \text{Form}(\mathcal{L})$ with $x_i \notin \text{Free}(B)$
(Exercise Sheet # 4, (2)(i))

MP $\Rightarrow \Delta \vdash (\exists x_i \psi \rightarrow \chi)$
and $\Delta \vdash (\exists x_i \psi \rightarrow \neg\chi)$
($\chi, \neg\chi \in \text{Sent}(\mathcal{L})$, so $x_i \notin \text{Free}(\chi)$)

By hypothesis, $\Delta \vdash \exists x_i \psi$
 $\Rightarrow$ by MP, $\Delta \vdash \chi$ and $\Delta \vdash \neg\chi$
contradicting consistency of Δ .

□_{13.6}

Proof of 13.4:

Let Δ be any consistent set of sentences.

to show: Δ has a model assuming that any maximal consistent, witnessing set of sentences has a model.

By 13.3(a), Δ' is consistent and does not contain any c_{2m+1} .

Let $\phi_1, \phi_2, \phi_3, \dots$ be an enumeration of $\text{Sent}(\mathcal{L}' \cup \{c_1, c_3, c_5, \dots\})$.

Construct finite sets $\subseteq \text{Sent}(\mathcal{L}' \cup \{c_1, c_3, c_5, \dots\})$

$$\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \dots$$

such that $\Delta' \cup \Gamma_n$ is consistent for each $n \geq 0$ as follows:

Let $\Gamma_0 := \emptyset$.

If Γ_n has been constructed let

$$\Gamma_{n+1/2} := \begin{cases} \Gamma_n \cup \{\phi_{n+1}\} & \text{if } \Delta' \cup \Gamma_n \cup \{\phi_{n+1}\} \\ & \text{is consistent} \\ \Gamma_n \cup \{\neg\phi_{n+1}\} & \text{otherwise} \end{cases}$$

$\Rightarrow \Gamma_{n+1/2}$ is consistent (Lemma 13.5)

Now, if $\neg\phi_{n+1} \in \Gamma_{n+1/2}$ or if ϕ_{n+1} is *not* of the form $\exists x_i \psi$, let $\Gamma_{n+1} := \Gamma_{n+1/2}$.

If not, i.e. if $\phi_{n+1} = \exists x_i \psi \in \Gamma_{n+1/2}$ then $\Delta' \cup \Gamma_{n+1/2} \vdash \exists x_i \psi$.

Choose m large enough such that c_{2m+1} does not occur in any formula in $\Delta' \cup \Gamma_{n+1/2} \cup \{\psi\}$ (possible since $\Gamma_{n+1/2} \cup \{\psi\}$ is finite and Δ' has only even constants).

Let $\Gamma_{n+1} := \Gamma_{n+1/2} \cup \{\psi[c_{2m+1}/x_i]\}$
 $\Rightarrow$ by Lemma 13.6, Γ_{n+1} is consistent.

Let $\Gamma := \Delta' \cup \bigcup_{n \geq 0} \Gamma_n$.

$\Rightarrow \Gamma$ is maximal consistent
(as in Propositional Calculus)
and Γ is witnessing (by construction).

By assumption, Γ has a model, say $\mathcal{A}$.

$\Rightarrow$ in particular, $\Gamma \models \delta$ for any $\delta \in \Delta'$

$\Rightarrow$ by Lemma 13.3(b), Δ has a model

□_{13.4}