

14. Applications of Gödel's Completeness Theorem

14.1 Compactness Theorem for Predicate Calculus

Let $\mathcal{L}$ be a first-order language and let $\Gamma \subseteq \text{Sent}(\mathcal{L})$.

Then Γ has a model iff every finite subset of Γ has a model.

Proof: as for Propositional Calculus – Exercise sheet # 4, (5)(ii).

14.2 Example

Let $\Gamma \subseteq \text{Sent}(\mathcal{L})$. Assume that for every $N \geq 1$, Γ has a model whose domain has at least N elements.

Then Γ has a model with an infinite domain.

Proof:

For each $n \geq 2$ let χ_n be the sentence

$$\exists x_1 \exists x_2 \cdots \exists x_n \bigwedge_{1 \leq i < j \leq n} \neg x_i \doteq x_j$$

$\Rightarrow$ for any $\mathcal{L}$ -structure $\mathcal{A} = \langle A; \dots \rangle$,

$$\mathcal{A} \models \chi_n \text{ iff } \#A \geq n$$

Let $\Gamma' := \Gamma \cup \{\chi_n \mid n \geq 1\}$.

If $\Gamma_0 \subseteq \Gamma'$ is finite,

let N be maximal with $\chi_N \in \Gamma_0$.

By hypothesis, $\Gamma \cup \{\chi_N\}$ has a model.

$\Rightarrow \Gamma_0$ has a model

(note that $\vdash \chi_N \rightarrow \chi_{N-1} \rightarrow \chi_{N-2} \rightarrow \dots$)

$\Rightarrow$ By the Compactness Theorem 14.1,

Γ' has a model, say $\mathcal{A} = \langle A; \dots \rangle$

$\Rightarrow \mathcal{A} \models \chi_n$ for all $n \Rightarrow \#A = \infty$ $\square$

14.3 The Löwenheim-Skolem Theorem

Let $\Gamma \subseteq \text{Sent}(\mathcal{L})$ be consistent.

Then Γ has a model with a countable domain.

Proof:

This follows from the proof of the Completeness Theorem:

The **term model** constructed there was countable, because there are only countably many closed terms.

□

14.4 Definition

(i) Let $\mathcal{A}$ be an $\mathcal{L}$ -structure.

Then the $\mathcal{L}$ -**theory of $\mathcal{A}$** is

$$\text{Th}(\mathcal{A}) := \{\phi \in \text{Sent}(\mathcal{L}) \mid \mathcal{A} \models \phi\},$$

the set of all $\mathcal{L}$ -sentences true in $\mathcal{A}$.

Note: $\text{Th}(\mathcal{A})$ is maximal consistent.

(ii) If $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{L}$ -structures with $\text{Th}(\mathcal{A}) = \text{Th}(\mathcal{B})$ then $\mathcal{A}$ and $\mathcal{B}$ are **elementarily equivalent** (in symbols ' $\mathcal{A} \equiv \mathcal{B}$ ').

14.5 Remark

Let $\Gamma \subseteq \text{Sent}(\mathcal{L})$ be any set of $\mathcal{L}$ -sentences.
Then TFAE:

- (i) Γ is strongly maximal consistent (i.e. for each $\mathcal{L}$ -sentence ϕ , $\phi \in \Gamma$ or $\neg\phi \in \Gamma$)
- (ii) $\Gamma = \text{Th}(\mathcal{A})$ for some $\mathcal{L}$ -structure $\mathcal{A}$

Proof:

(i) $\Rightarrow$ (ii): Completeness Theorem

Rest: clear. □

Note that Γ is maximal consistent if and only if Γ has models, and, for any two models $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \equiv \mathcal{B}$.

A worked example: Dense linear orderings without endpoints

Let $\mathcal{L} = \{<\}$ be the language with just one binary predicate symbol ' $<$ ', and let Γ be the $\mathcal{L}$ -theory of dense linear orderings without endpoints (cf. Example 10.8) consisting of the axioms $\psi_1, \dots, \psi_4$:

$$\begin{aligned}\psi_1 : & \quad \forall x \forall y ((x < y \vee x \dot{=} y \vee y < x) \\ & \quad \wedge \neg((x < y \wedge x \dot{=} y) \vee (x < y \wedge y < x))) \\ \psi_2 : & \quad \forall x \forall y \forall z (x < y \wedge y < z \rightarrow x < z) \\ \psi_3 : & \quad \forall x \forall z (x < z \rightarrow \exists y (x < y \wedge y < z)) \\ \psi_4 : & \quad \forall y \exists x \exists z (x < y \wedge y < z)\end{aligned}$$

14.6 (a) Examples

$\mathbf{Q}$, $\mathbf{R}$, $]0, 1[$, $\mathbf{R} \setminus \{0\}$, $[\sqrt{2}, \pi] \cap \mathbf{Q}$, $]0, 1[\cup]2, 3[$,
or $\mathbf{Z} \times \mathbf{R}$ with lexicographic ordering:

$$(a, b) < (c, d) \Leftrightarrow a < c \text{ or } (a = c \ \& \ b < d)$$

(b) Counterexamples $[0, 1]$, $\mathbf{Z}$, $\{0\}$, $\mathbf{R} \setminus]0, 1[$
or $\mathbf{R} \times \mathbf{Z}$ with lexicographic ordering

14.7 Theorem

Let Γ be the theory of dense linear orderings without endpoints, and let $\mathcal{A} = \langle A; <_{\mathcal{A}} \rangle$ and $\mathcal{B} = \langle B; <_{\mathcal{B}} \rangle$ be two countable models.

Then $\mathcal{A}$ and $\mathcal{B}$ are isomorphic, i.e. there is an order preserving bijection between A and B .

Proof: Note: A and B are infinite.

Choose an enumeration (no repeats)

$$\begin{aligned} A &= \{a_1, a_2, a_3, \dots\} \\ B &= \{b_1, b_2, b_3, \dots\} \end{aligned}$$

Define $\phi : A \rightarrow B$ recursively s.t. for all n :

$$(\star_n) \text{ for all } i, j \leq n : \phi(a_i) <_{\mathcal{B}} \phi(a_j) \Leftrightarrow a_i <_{\mathcal{A}} a_j$$

Suppose ϕ has been defined on $\{a_1, \dots, a_n\}$ satisfying $(\star_n)$.

Let $\phi(a_{n+1}) = b_m$,
where $m > 1$ is minimal s.t.

for all $i \leq n : b_m <_{\mathcal{B}} \phi(a_i) \Leftrightarrow a_{n+1} <_{\mathcal{A}} a_i$,

i.e. the position of $\phi(a_{n+1})$
relative to $\phi(a_1), \dots, \phi(a_n)$

is the same as that of a_{n+1}
relative to $a_1, \dots, a_n$

(possible as $\mathcal{A}, \mathcal{B} \models \Gamma$).

$\Rightarrow (\star_{n+1})$ holds for $a_1, \dots, a_{n+1}$

$\Rightarrow \phi$ is injective

And ϕ is surjective, by minimality of m . $\square$

14.8 Corollary

Γ is maximal consistent

Proof:

to show: $\text{Th}(\mathcal{A}) = \text{Th}(\mathcal{B})$ for any $\mathcal{A}, \mathcal{B} \models \Gamma$
(by Remark 14.5)

By the Theorem of Löwenheim-Skolem (14.3),
 $\text{Th}(\mathcal{A})$ and $\text{Th}(\mathcal{B})$ have countable models,
say $\mathcal{A}_0$ and $\mathcal{B}_0$.

$\Rightarrow \text{Th}(\mathcal{A}_0) = \text{Th}(\mathcal{A})$ and $\text{Th}(\mathcal{B}_0) = \text{Th}(\mathcal{B})$

Theorem 14.7 $\Rightarrow \mathcal{A}_0$ and $\mathcal{B}_0$ are isomorphic

$\Rightarrow \text{Th}(\mathcal{A}_0) = \text{Th}(\mathcal{B}_0)$

$\Rightarrow \text{Th}(\mathcal{A}) = \text{Th}(\mathcal{B})$

□

Recall that $\mathbf{R}$ is **complete**:

for any subsets $A, B \subseteq \mathbf{R}$ with $A' <' B$
(i.e. $a < b$ for any $a \in A, b \in B$)
there is $\gamma \in \mathbf{R}$ with $A' \leq' \{\gamma\}' \leq' B$.

$\mathbf{Q}$ is *not* complete:

$$\begin{aligned} \text{take } A &= \{x \in \mathbf{Q} \mid x < \pi\} \\ B &= \{x \in \mathbf{Q} \mid \pi < x\} \end{aligned}$$

14.9 Corollary

$$Th(\langle \mathbf{Q}; < \rangle) = Th(\langle \mathbf{R}; < \rangle)$$

In particular, the completeness of $\mathbf{R}$ is not a first-order property,

i.e. there is no $\Delta \subseteq \text{Sent}(\mathcal{L})$ such that for all $\mathcal{L}$ -structures $\langle A; < \rangle$,

$$\langle A; < \rangle \models \Delta \text{ iff } \langle A; < \rangle \text{ is complete.}$$