15. Normal Forms

(a) Prenex Normal Form

A formula is in **prenex normal form (PNF)** if it has the form

$$Q_1 x_{i_1} Q_2 x_{i_2} \cdots Q_r x_{i_r}\, \psi,$$

where each Q_i is a quantifier
(i.e. either $\forall$ or $\exists$), and where
ψ is a formula containing no quantifiers.

15.1 PNF-Theorem
Every $\phi \in$ Form($\mathcal{L}$) is logically equivalent to an $\mathcal{L}$-formula in **PNF**.

Proof: Induction on ϕ
(working in the language with $\forall, \exists, \neg, \wedge$):

ϕ atomic: OK

$\phi = \neg\psi$,
say $\phi \leftrightarrow \neg Q_1 x_{i_1} Q_2 x_{i_2} \cdots Q_r x_{i_r}\, \chi$

Then $\phi \leftrightarrow Q_1^- x_{i_1} Q_2^- x_{i_2} \cdots Q_r^- x_{i_r}\, \neg\chi$,
where $Q^- = \exists$ if $Q = \forall$, and $Q^- = \forall$ if $Q = \exists$

$\phi = (\chi \wedge \rho)$ with χ, ρ in PNF
Note that $\vdash (\forall x_j \psi[x_j/x_i] \leftrightarrow \forall x_i \psi)$,
provided x_j does not occur in ψ (Ex. 12.5)

So w.l.o.g. the variables quantified over in χ do not occur in ρ and vice versa.

But then, e.g. $(\forall x\alpha \wedge \exists y\beta) \leftrightarrow \forall x \exists y(\alpha \wedge \beta)$ etc.
$\square$

(b) Skolem Normal Form

Recall: In the proof of CT, we introduced witnessing new constants for existential formulas such that

$\exists x \phi(x)$ is satisfiable iff $\phi(c)$ is satisfiable.

This way an $\exists x$ in front of a formula could be removed at the expense of a new constant.

Now we remove existential quantifiers 'inside' a formula at the expense of extra function symbols:

15.2 Observation:
Let $\phi = \phi(x, y)$ be an $\mathcal{L}$-formula with $x, y \in$Free(ϕ). Let f be a new unary function symbol (not in $\mathcal{L}$).

Then $\forall x \exists y \phi(x,y)$ is satisfiable iff $\forall x \phi(x, f(x))$ is satisfiable.
(f is called a **Skolem function** *for ϕ.)*

Proof: '$\Leftarrow$': clear

'$\Rightarrow$': Let $\mathcal{A}$ be an $\mathcal{L}$-structure with $\mathcal{A} \models \forall x \exists y \phi(x,y)$

$\Rightarrow$ for every $a \in A$ there is some $b \in A$ with $\phi(a,b)$

Interpret f by a function assigning to each $a \in A$ one such b
(this uses the Axiom of Choice!). $\square$

Example: $\mathbf{R} \models \forall x \exists y (x \doteq y^2 \vee x \doteq -y^2)$ – here $f(x) = \sqrt{|\, x \,|}$ will do.

15.3 Theorem
*For every $\mathcal{L}$-formula ϕ
there is a formula $\phi^\star$
(with new constant and function symbols)
having only universal quantifiers in its PNF
such that*

$$\phi \text{ is satisfiable iff } \phi^\star \text{ is.}$$

*More precisely,
any $\mathcal{L}$-structure $\mathcal{A}$
can be made into a structure $\mathcal{A}^\star$
interpreting the new constant and function symbols
such that*

$$\mathcal{A} \models \phi \text{ iff } \mathcal{A}^\star \models \phi^\star.$$

Lecture 16 - 6/6