

6.6 The Deduction Theorem for L_0

*For any $\Gamma \subseteq \text{Form}(\mathcal{L}_0)$ and
for any $\alpha, \beta \in \text{Form}(\mathcal{L}_0)$:*

if $\Gamma \cup \{\alpha\} \vdash \beta$ then $\Gamma \vdash (\alpha \rightarrow \beta)$.

Proof:

We prove by induction on m :

if $\alpha_1, \dots, \alpha_m$ *is derivable in L_0
from the hypotheses $\Gamma \cup \{\alpha\}$*
then *for all $i \leq m$
 $(\alpha \rightarrow \alpha_i)$ is derivable in L_0
from the hypotheses Γ .*

m=1

Either α_1 is an Axiom or $\alpha_1 \in \Gamma \cup \{\alpha\}$.

Case 1: α_1 is an Axiom

Then

- | | | |
|---|--|-------------------|
| 1 | α_1 | [Axiom] |
| 2 | $(\alpha_1 \rightarrow (\alpha \rightarrow \alpha_1))$ | [Instance of A1] |
| 3 | $(\alpha \rightarrow \alpha_1)$ | [MP 1,2] |

is a derivation of $(\alpha \rightarrow \alpha_1)$ from hypotheses \emptyset .

Note that if $\Delta \vdash \psi$ and $\Delta \subseteq \Delta'$, then obviously $\Delta' \vdash \psi$.

Thus $(\alpha \rightarrow \alpha_1)$ is derivable in L_0 from hypotheses Γ .

Case 2: $\alpha_1 \in \Gamma \cup \{\alpha\}$

If $\alpha_1 \in \Gamma$ then same proof as above works (with justification on line 1 changed to ' $\in \Gamma$ ').

If $\alpha_1 = \alpha$, then, by Example 6.3, $\vdash (\alpha \rightarrow \alpha_1)$, hence $\Gamma \vdash (\alpha \rightarrow \alpha_1)$.

Induction Step

IH: Suppose result is true for derivations of length $\leq m$.

Let $\alpha_1, \dots, \alpha_{m+1}$ be a derivation in L_0 from $\Gamma \cup \{\alpha\}$.

Then **either** α_{m+1} is an axiom or $\alpha_{m+1} \in \Gamma \cup \{\alpha\}$ – in these cases proceed as above, even without IH.

Or α_{m+1} is obtained by MP from some earlier α_j, α_k , i.e. there are $j, k < m + 1$ such that $\alpha_j = (\alpha_k \rightarrow \alpha_{m+1})$.

By IH, we have

$$\begin{array}{ll} & \Gamma \vdash (\alpha \rightarrow \alpha_k) \\ \text{and} & \Gamma \vdash (\alpha \rightarrow \alpha_j), \\ \text{so} & \Gamma \vdash (\alpha \rightarrow (\alpha_k \rightarrow \alpha_{m+1})) \end{array}$$

Let β_1, \dots, β_r be a derivation in L_0 of $(\alpha \rightarrow \alpha_k) = \beta_r$ from Γ

and let $\gamma_1, \dots, \gamma_s$ be a derivation in L_0 of $(\alpha \rightarrow (\alpha_k \rightarrow \alpha_{m+1})) = \gamma_s$ from Γ .

Then

| | | |
|----------|---|--------------------|
| 1 | β_1 | |
| \vdots | \vdots | |
| $r-1$ | β_{r-1} | |
| r | $(\alpha \rightarrow \alpha_k)$ | |
| $r+1$ | γ_1 | |
| \vdots | \vdots | |
| $r+s-1$ | γ_{s-1} | |
| $r+s$ | $(\alpha \rightarrow (\alpha_k \rightarrow \alpha_{m+1}))$ | |
| $r+s+1$ | $((\alpha \rightarrow (\alpha_k \rightarrow \alpha_{m+1})) \rightarrow$ $((\alpha \rightarrow \alpha_k) \rightarrow (\alpha \rightarrow \alpha_{m+1})))$ | [A2] |
| $r+s+2$ | $((\alpha \rightarrow \alpha_k) \rightarrow (\alpha \rightarrow \alpha_{m+1}))$ | [MP $r+s, r+s+1$] |
| $r+s+3$ | $(\alpha \rightarrow \alpha_{m+1})$ | [MP $r, r+s+2$] |

is a derivation of $(\alpha \rightarrow \alpha_{m+1})$ in L_0 from Γ . \square

6.7 Remarks

- Only needed instances of A1, A2 and the rule MP.
So any system that includes A1, A2 and MP satisfies the Deduction Theorem.
- Proof gives a precise **algorithm** for converting any derivation showing $\Gamma \cup \{\alpha\} \vdash \beta$ into one showing $\Gamma \vdash (\alpha \rightarrow \beta)$.
- Converse is easy:

If $\Gamma \vdash (\alpha \rightarrow \beta)$ then $\Gamma \cup \{\alpha\} \vdash \beta$.

Proof:

| | | |
|----------|----------------------------|--------------------------------|
| \vdots | \vdots | derivation from Γ |
| r | $\alpha \rightarrow \beta$ | |
| $r+1$ | α | $[\in \Gamma \cup \{\alpha\}]$ |
| $r+2$ | β | $[\text{MP } r, r+1]$ |

□

6.8 Example of use of DT

If $\Gamma \vdash (\alpha \rightarrow \beta)$ and $\Gamma \vdash (\beta \rightarrow \gamma)$
then $\Gamma \vdash (\alpha \rightarrow \gamma)$.

Proof:

By the deduction theorem ('DT'), it suffices
to show that $\Gamma \cup \{\alpha\} \vdash \gamma$.

| | | |
|----------|------------------------------|--------------------------------|
| \vdots | \vdots | proof from Γ |
| r | $(\alpha \rightarrow \beta)$ | |
| $r+1$ | \vdots | |
| \vdots | \vdots | proof from Γ |
| $r+s$ | $(\beta \rightarrow \gamma)$ | |
| $r+s+1$ | α | $[\in \Gamma \cup \{\alpha\}]$ |
| $r+s+2$ | β | $[\text{MP } r, r+s+1]$ |
| $r+s+3$ | γ | $[\text{MP } r+s, r+s+2]$ |

□

From now on we may treat DT as an additional
inference rule in L_0 .

6.9 Definition

The **sequent calculus** SQ is the system where a **proof** (or **derivation**) of $\phi \in \text{Form}(\mathcal{L}_0)$ from $\Gamma \subseteq \text{Form}(\mathcal{L}_0)$ is a finite sequence of **sequents**, i.e. of expressions of the form

$$\Delta \vdash_{SQ} \psi$$

with $\Delta \subseteq \text{Form}(\mathcal{L}_0)$ and $\Gamma \vdash_{SQ} \phi$ as last sequent.

Sequents may be formed according to the following rules

Ass: if $\psi \in \Delta$ then infer $\Delta \vdash_{SQ} \psi$

MP: from $\Delta \vdash_{SQ} \psi$ and $\Delta' \vdash_{SQ} (\psi \rightarrow \chi)$
infer $\Delta \cup \Delta' \vdash_{SQ} \chi$

DT: from $\Delta \cup \{\psi\} \vdash_{SQ} \chi$ infer $\Delta \vdash_{SQ} (\psi \rightarrow \chi)$

PC: from $\Delta \cup \{\neg\psi\} \vdash_{SQ} \chi$ and
 $\Delta' \cup \{\neg\psi\} \vdash_{SQ} \neg\chi$ infer $\Delta \cup \Delta' \vdash_{SQ} \psi$

‘PC’ stands for *proof by contradiction*

Note: no axioms.

6.10 Example of a proof in SQ

- | | | |
|---|---|----------|
| 1 | $\neg\beta \vdash_{SQ} \neg\beta$ | [Ass] |
| 2 | $(\neg\beta \rightarrow \neg\alpha) \vdash_{SQ} (\neg\beta \rightarrow \neg\alpha)$ | [Ass] |
| 3 | $(\neg\beta \rightarrow \neg\alpha), \neg\beta \vdash_{SQ} \neg\alpha$ | [MP 1,2] |
| 4 | $\alpha, \neg\beta \vdash_{SQ} \alpha$ | [Ass] |
| 5 | $(\neg\beta \rightarrow \neg\alpha), \alpha \vdash_{SQ} \beta$ | [PC 3,4] |
| 6 | $(\neg\beta \rightarrow \neg\alpha) \vdash_{SQ} (\alpha \rightarrow \beta)$ | [DT 5] |
| 7 | $\vdash_{SQ} ((\neg\beta \rightarrow \neg\alpha) \rightarrow (\alpha \rightarrow \beta))$ | [DT 6] |

So $\vdash_{SQ} A3$.

We'd better write ' $\Gamma \vdash_{L_0} \phi$ ' for ' $\Gamma \vdash \phi$ in L_0 '.

6.11 Theorem

L_0 and SQ are equivalent: for all Γ, ϕ

$$\Gamma \vdash_{L_0} \phi \text{ iff } \Gamma \vdash_{SQ} \phi.$$

Proof: Exercise