

11. Substitution

Goal: Given $\phi \in \text{Form}(\mathcal{L})$ and $x_i \in \text{Free}(\phi)$

- want to replace x_i by a term t to obtain a new formula $\phi[t/x_i]$

(read: ' ϕ with x_i replaced by t ')

- should have $\{\forall x_i \phi\} \models \phi[t/x_i]$

11.1 Example

Let $\mathcal{L} = \{f; c\}$ and let ϕ be $\exists x_1 f(x_1) \doteq x_0$.

$\Rightarrow \text{Free}(\phi) = \{x_0\}$ and

' $\forall x_0 \phi$ ', i.e. ' $\forall x_0 \exists x_1 f(x_1) \doteq x_0$ '

says that f is onto.

- if $t = c$ then $\phi[t/x_0]$ is $\exists x_1 f(x_1) \doteq c$

- but if $t = x_1$ then $\phi[t/x_0]$ is $\exists x_1 f(x_1) \doteq x_1$,
stating the existence of a fixed point of f —
no good: there are fixed point free onto functions, e.g. ' $+1$ ' on $\mathbb{Z}$.

Problem: the variable x_1 in t has become unintentionally bound in the substitution.

To avoid this we define:

11.2 Definition

For $\phi \in \text{Form}(\mathcal{L})$, for any variable x_i (not necessarily in $\text{Free}(\phi)$) and for any term $t \in \text{Term}(\mathcal{L})$, define the phrase

' t is free for x_i in ϕ '

and the substitution

$\phi[t/x_i]$ (' ϕ with x_i replaced by t ')

recursively as follows:

(i) if ϕ is atomic, then t is free for x_i in ϕ and $\phi[t/x_i]$ is the result of replacing every occurrence of x_i in ϕ by t .

(ii) if $\phi = \neg\psi$ then

t is free for x_i in ϕ iff t is free for x_i in ψ .

In this case, $\phi[t/x_i] = \neg\alpha$, where $\alpha = \psi[t/x_i]$.

(iii) if $\phi = (\psi \rightarrow \chi)$ then

t is free for x_i in ϕ iff

t is free for x_i in both ψ and χ .

In this case, $\phi[t/x_i] = (\alpha \rightarrow \beta)$,

where $\alpha = \psi[t/x_i]$ and $\beta = \chi[t/x_i]$.

(iv) if $\phi = \forall x_j \psi$ then

t is free for x_i in ϕ

if $i = j$ or

if $i \neq j$, and x_j does not occur in t ,
and t is free for x_i in ψ .

In this case $\phi[t/x_i] = \begin{cases} \phi & \text{if } i = j \\ \forall x_j \alpha & \text{if } i \neq j, \end{cases}$

where $\alpha = \psi[t/x_i]$.

11.3 Example

Let $\mathcal{L} = \{f, g\}$ and let ϕ be $\exists x_1 f(x_1) \doteq x_0$.

$\Rightarrow g(x_0, x_2)$ is free for x_0 in ϕ

and $\phi[g(x_0, x_2)/x_0]$ is $\exists x_1 f(x_1) \doteq g(x_0, x_2)$,

but $g(x_0, x_1)$ is *not* free for x_0 in ϕ .

11.4 Lemma

Let $\mathcal{L}$ be a first-order language, $\mathcal{A}$ an $\mathcal{L}$ -structure, $\phi \in \text{Form}(\mathcal{L})$ and t a term free for the variable x_i in ϕ . Let v be an assignment in $\mathcal{A}$ and define

$$v'(x_j) := \begin{cases} v(x_j) & \text{if } j \neq i \\ \tilde{v}(t) & \text{if } j = i \end{cases}$$

Then $\mathcal{A} \models \phi[v']$ iff $\mathcal{A} \models \phi[t/x_i][v]$.

Proof: **1.** For $u \in \text{Term}(\mathcal{L})$ let

$u[t/x_i] :=$ the term obtained by replacing
each occurrence of x_i in u by t

$\Rightarrow \tilde{v}'(u) = \tilde{v}(u[t/x_i])$
(Exercise)

2. If ϕ is **atomic**, say

$\phi = P(t_1, \dots, t_k)$ for some $P = P_i^{(k)} \in \text{Pred}(\mathcal{L})$
then

$$\mathcal{A} \models \phi[v']$$

$$\text{iff } P_{\mathcal{A}}(\tilde{v}'(t_1), \dots, \tilde{v}'(t_k)) \quad \text{by def. '}\models\text{'}$$

$$\text{iff } P_{\mathcal{A}}(\tilde{v}(t_1[t/x_i]), \dots, \tilde{v}(t_k[t/x_i])) \quad \text{by 1.}$$

$$\text{iff } \mathcal{A} \models P(t_1[t/x_i], \dots, t_k[t/x_i])[v] \quad \text{by def. '}\models\text{'}$$

$$\text{iff } \mathcal{A} \models \phi[t/x_i][v]$$

Similarly, if ϕ is $t_1 \doteq t_2$.

3. Induction step

The cases $\neg$ and $\rightarrow$ are routine.

$\leadsto$ the only interesting case is $\phi = \forall x_j \psi$.

IH: Lemma holds for ψ .

Case 1: $j = i$

$\Rightarrow \phi[t/x_i] = \phi$ by Definition 11.2.(iv)

$x_i = x_j \notin \text{Free}(\phi)$

$\Rightarrow v$ and v' agree on all $x \in \text{Free}(\phi)$

$\Rightarrow$ by Lemma 10.3,

$$\mathcal{A} \models \phi[v'] \text{ iff } \mathcal{A} \models \phi[v] \text{ iff } \mathcal{A} \models \phi[t/x_i][v]$$

Case 2: $j \neq i$

‘ $\Rightarrow$ ’: Suppose $\mathcal{A} \models \forall x_j \psi[v']$ (★)

to show: $\mathcal{A} \models \forall x_j \psi[t/x_i][v]$

So let $v^\star$ agree with v except possibly at x_j .
to show: $\mathcal{A} \models \psi[t/x_i][v^\star]$

Define $v^{\star'}(x_k) := \begin{cases} v^\star(x_k) & \text{if } k \neq i \\ \widetilde{v^\star}(t) & \text{if } k = i \end{cases}$

t is free for x_i in $\phi \Rightarrow$

t is free for x_i in ψ and t does not contain x_j .

IH $\Rightarrow$ enough to show: $\mathcal{A} \models \psi[v^{\star'}]$

$v^{\star'}$ and v' agree except possibly at x_i and x_j .

But, in fact, they *do* agree at x_i :

$$v'(x_i) = \widetilde{v}(t) = \widetilde{v^\star}(t) = v^{\star'}(x_i),$$

where the 2nd equality holds, because v and $v^\star$ agree except possibly at x_i , which does not occur in t .

So $v^{\star'}$ and v' agree except possibly at x_j

$\Rightarrow$ by $(\star)$, $\mathcal{A} \models \psi[v^{\star'}]$ as required.

‘ $\Leftarrow$ ’: similar. □

11.5 Corollary

For any $\phi \in \text{Form}(\mathcal{L})$, $t \in \text{Term}(\mathcal{L})$,

$$\models (\forall x_i \phi \rightarrow \phi[t/x_i]),$$

provided that the term t is free for x_i in ϕ .

Proof: Let $\mathcal{A}$ be an $\mathcal{L}$ -structure and let v be an assignment in $\mathcal{A}$.

Assume $\mathcal{A} \models \forall x_i \phi[v]$ (★)

to show: $\mathcal{A} \models \phi[t/x_i][v]$

By Lemma 11.4, it suffices to show $\mathcal{A} \models \phi[v']$, where

$$v'(x_j) := \begin{cases} v(x_j) & \text{for } j \neq i \\ \tilde{v}(t) & \text{for } j = i. \end{cases}$$

Since v and v' agree except possibly at x_i , this follows from (★).

□