

Problem Sheet # 4

(1)(i) Let \mathcal{L} be a first-order language, let \mathcal{A} be an \mathcal{L} -structure, let v be an assignment in \mathcal{A} and let u and t be terms in \mathcal{L} . Define a new assignment v' by

$$v'(x_j) := \begin{cases} v(x_j) & \text{if } j \neq i \\ \tilde{v}(t) & \text{if } j = i \end{cases}$$

Let $u[t/x_i]$ be the term obtained by replacing each occurrence of x_i in u by t . Show that then $\tilde{v}'(u) = \tilde{v}(u[t/x_i])$.

(ii) Prove that for any closed (i.e. variable free) terms t_1, t_2, t_3 one has $\vdash (t_1 \doteq t_2 \rightarrow t_2 \doteq t_1)$ and $\{t_1 \doteq t_2, t_2 \doteq t_3\} \vdash t_1 \doteq t_3$.

(2)(i) Prove that $\vdash (\forall x_i(A \rightarrow B) \rightarrow (\exists x_i A \rightarrow B))$ for any formulas A, B provided that the variable x_i does not occur free in B .

(ii) Let ϕ be a formula with just one variable, x_i say, occurring free and let Δ be a set of sentences. Assume that the constant symbol c_j does not occur in ϕ nor in any sentence in Δ , and that $\Delta \vdash \phi[c_j/x_i]$. Sketch a proof that $\Delta \vdash \phi$.

[Hint: First reduce to the case that Δ is finite and choose m so large that the variable x_m does not occur in Δ nor in any formula in a derivation of $\phi[c_j/x_i]$ from hypotheses Δ . Then change every occurrence of c_j to x_m .]

(3) Derive the following theorems:

(i) If x_i does not occur free in A then

(a) $\vdash (\exists x_i(A \rightarrow B) \rightarrow (A \rightarrow \exists x_i B))$,

(b) $\vdash ((A \rightarrow \exists x_i B) \rightarrow \exists x_i(A \rightarrow B))$.

(ii) If the only variables occurring free in ϕ are x_i and x_j (where $i \neq j$) then

(a) $\vdash (\forall x_i \neg \phi \rightarrow \neg \forall x_j \phi)$

(b) $\vdash (\exists x_i \forall x_j \phi \rightarrow \forall x_j \exists x_i \phi)$.

p.t.o.

(4) Let f, g be binary function symbols, P a binary predicate symbol, c, d constant symbols and let $\mathcal{L} := \{f, g; P; c, d\}$.

Consider $\mathcal{R} := \langle \mathbb{R}; +, \cdot; <; 0, 1 \rangle$ as \mathcal{L} -structure. Let h be a unary function symbol, let $\mathcal{L}' := \mathcal{L} \cup \{h\}$ and let \mathcal{R}' be \mathcal{R} together with an interpretation $h_{\mathcal{R}'}$ of h in \mathcal{R} .

Find \mathcal{L}' -formulas ϕ and ψ such that

- (i) $\mathcal{R}' \models \phi$ iff $h_{\mathcal{R}'}$ is continuous.
- (ii) $\mathcal{R}' \models \psi$ iff $h_{\mathcal{R}'}$ is differentiable.

(5)(i) Let $\mathcal{L} := \{+, \cdot; 0, 1\}$ be the language of rings (i.e. $+$ a binary function symbol etc.). Write down sets of formulas Φ_p (for p a prime or $p = 0$) whose models are exactly all fields of characteristic p .

(ii) State the Compactness Theorem for sets of sentences and show how it follows from the Soundness and Completeness Theorems.

(iii) Prove that Φ_0 in (i) cannot be chosen finite.

(6)(i) Axiomatize the first-order theory Σ of ordered fields in the language $\mathcal{L} := \{+, \cdot; <; 0, 1\}$.

(ii) Which of the following is a model of Σ :

- (α) \mathbb{Q} with the usual interpretations
- (β) \mathbb{R} with the usual interpretations
- (γ) \mathbb{C} with $a + bi < c + di$ iff $a^2 + b^2 < c^2 + d^2$
- (δ) \mathbb{F}_p , the field with p (prime) elements with $0 < 1 < 2 < \dots < p - 1$.

(iii) Is Σ consistent? Is it maximal consistent?

(iv) Recall that the ordering on \mathbb{Q} resp. on \mathbb{R} is *archimedean*, i.e. for every $x \in \mathbb{Q}$ resp. $x \in \mathbb{R}$ there is some $n \in \mathbb{N}$ with $-n < x < n$. Use the Compactness Theorem to prove that archimedeanity is not a first-order property. (Hint: introduce a new constant symbol c .)