

10. Free and bound variables

Recall Example 9.5: The formula

$$\phi = \forall x_0 \forall x_1 (f(x_0, x_2) \doteq f(x_1, x_2) \rightarrow x_0 \doteq x_1)$$

- is true in $\langle \mathbf{Z}; \cdot \rangle$ under any assignment v with $v(x_2) = 2$
- but false when $v(x_2) = 0$.

Whether or not $\mathcal{A} \models \phi[v]$ only depends on $v(x_2)$, not on $v(x_0)$ or $v(x_1)$.

say: the occurrence of x_2 in ϕ is **free**.

10.1 Definition

Let $\mathcal{L}$ be a first-order language, ϕ an $\mathcal{L}$ -formula, and $x \in \{x_0, x_1, \dots\}$ a variable occurring in ϕ .

The occurrence of x in ϕ is **free**, if

- (i) ϕ is atomic, or
- (ii) $\phi = \neg\psi$ resp. $\phi = (\chi \rightarrow \rho)$ and x occurs free in ψ resp. in χ or ρ , or
- (iii) $\phi = \forall x_i \psi$, x occurs free in ψ , and $x \neq x_i$.

Every other occurrence of x in ϕ is called **bound**.

In particular, if $x = x_i$ and $\phi = \forall x_i \psi$, then x is bound in ϕ .

10.2 Example

$$(\exists x_0 P(\underbrace{x_0}_b, \underbrace{x_1}_f) \vee \forall x_1 (P(\underbrace{x_0}_f, \underbrace{x_1}_b) \rightarrow \exists x_0 P(\underbrace{x_0}_b, \underbrace{x_1}_b)))$$

10.3 Lemma

Let $\mathcal{L}$ be a language, let $\mathcal{A}$ be an $\mathcal{L}$ -structure, let v, v' be assignments in $\mathcal{A}$ and let ϕ be an $\mathcal{L}$ -formula.

Suppose $v(x_i) = v'(x_i)$ for every variable x_i with a free occurrence in ϕ .

Then

$$\mathcal{A} \models \phi[v] \text{ iff } \mathcal{A} \models \phi[v'].$$

Proof:

For ϕ atomic: exercise

Now use induction on the length of ϕ :

- $\phi = \neg\psi$ and $\phi = (\chi \rightarrow \rho)$: easy
- $\phi = \forall x_i \psi$:

IH: Assume the Lemma holds for ψ .

Let

$\text{Free}(\phi) := \{x_j \mid x_j \text{ occurs free in } \phi\}$

$\text{Free}(\psi) := \{x_j \mid x_j \text{ occurs free in } \psi\}$

$\Rightarrow x_i \notin \text{Free}(\phi)$ and

$$\text{Free}(\phi) = \text{Free}(\psi) \setminus \{x_i\}$$

Assume $\mathcal{A} \models \forall x_i \psi[v]$ (★)

to show: for any v^* agreeing with v' except possibly at x_i : $\mathcal{A} \models \psi[v^*]$.

for all $x_j \in \text{Free}(\phi)$:

$$v^*(x_j) = v(x_j) = v'(x_j).$$

Let $v^+(x_j) := \begin{cases} v(x_j) & \text{if } j \neq i \\ v^*(x_j) & \text{if } j = i \end{cases}$

Then v^+ agrees with v except possibly at x_i .

Hence, by (★), $\mathcal{A} \models \psi[v^+]$.

But $v^*(x_j) = v^+(x_j)$ for all $x_j \in \text{Free}(\psi)$.

$\Rightarrow$ by IH, $\mathcal{A} \models \psi[v^*]$

□

10.4 Corollary

Let $\mathcal{L}$ be a language, $\alpha, \beta \in \text{Form}(\mathcal{L})$. Assume the variable x_i has no free occurrence in α . Then

$$\models (\forall x_i(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \forall x_i\beta)).$$

Proof:

Let $\mathcal{A}$ be an $\mathcal{L}$ -structure and let v be an assignment in $\mathcal{A}$ such that

$$\mathcal{A} \models \forall x_i(\alpha \rightarrow \beta)[v] \quad (\star)$$

to show: $\mathcal{A} \models (\alpha \rightarrow \forall x_i\beta)[v]$.

So suppose $\mathcal{A} \models \alpha[v]$

to show: $\mathcal{A} \models \forall x_i\beta[v]$.

So let v^* be an assignment agreeing with v except possibly at x_i .

We want: $\mathcal{A} \models \beta[v^*]$

x_i is not free in $\alpha \Rightarrow_{10.3} \mathcal{A} \models \alpha[v^*]$

$(\star) \Rightarrow \mathcal{A} \models (\alpha \rightarrow \beta)[v^*]$

$\Rightarrow \mathcal{A} \models \beta[v^*]$

□

10.5 Definition

A formula ϕ without free (occurrence of) variables is called a **statement** or a **sentence**.

So for any $\mathcal{L}$ -structure $\mathcal{A}$ and any assignment v in $\mathcal{A}$, whether or not $\mathcal{A} \models \phi[v]$ does not depend on v .

So we write $\mathcal{A} \models \phi$ if $\mathcal{A} \models \phi[v]$ for some/all v .

Say: ϕ is **true** in $\mathcal{A}$, or $\mathcal{A}$ is a **model** of ϕ .

($\rightsquigarrow$ 'Model Theory')

10.6 Example

Let $\mathcal{L} = \{f, c\}$ be a language, where f is a binary function symbol, and c is a constant symbol.

Consider the sentences (we write x, y, z instead of x_0, x_1, x_2)

$$\phi_1 : \forall x \forall y \forall z f(x, f(y, z)) \doteq f(f(x, y), z)$$

$$\phi_2 : \forall x \exists y (f(x, y) \doteq c \wedge f(y, x) \doteq c)$$

$$\phi_3 : \forall x (f(x, c) \doteq x \wedge f(c, x) \doteq x)$$

and let $\phi = \phi_1 \wedge \phi_2 \wedge \phi_3$.

Let $\mathcal{A} = \langle A; \circ; e \rangle$ be an $\mathcal{L}$ -structure (i.e. $\circ$ is an interpretation of f , and e is an interpretation of c .)

Then $\mathcal{A} \models \phi$ iff $\mathcal{A}$ is a group.

10.7 Example

Let $\mathcal{L} = \{E\}$ be a language with $E = P_i^{(2)}$ a binary relation symbol. Consider

$$\chi_1 : \forall x E(x, x)$$

$$\chi_2 : \forall x \forall y (E(x, y) \leftrightarrow E(y, x))$$

$$\chi_3 : \forall x \forall y \forall z (E(x, y) \rightarrow (E(y, z) \rightarrow E(x, z)))$$

Then for any $\mathcal{L}$ -structure $\langle A; R \rangle$:

$\langle A; R \rangle \models (\chi_1 \wedge \chi_2 \wedge \chi_3)$ iff

R is an equivalence relation on A .

Note: Most mathematical concepts can be captured by first-order formulas.

10.8 Example

Let P be a 2-place (i.e. binary) predicate symbol, $\mathcal{L} := \{P\}$. Consider the statements

$$\begin{aligned}\psi_1 : & \forall x \forall y (P(x, y) \vee x \dot{=} y \vee P(y, x)) \\ & (\vee \text{ means } \textit{either - or exclusively:} \\ & (\alpha \vee \beta) :\Leftrightarrow ((\alpha \vee \beta) \wedge \neg(\alpha \wedge \beta))) \\ \psi_2 : & \forall x \forall y \forall z ((P(x, y) \wedge P(y, z)) \rightarrow P(x, z)) \\ \psi_3 : & \forall x \forall z (P(x, z) \rightarrow \exists y (P(x, y) \wedge P(y, z))) \\ \psi_4 : & \forall y \exists x \exists z (P(x, y) \wedge P(y, z))\end{aligned}$$

These are the axioms for a **dense linear order without endpoints**. Let $\psi = (\psi_1 \wedge \dots \wedge \psi_4)$. Then $\langle \mathbf{Q}; <> \models \psi$ and $\langle \mathbf{R}; <> \models \psi$.

But: ‘Completeness’ of $\langle \mathbf{R}; <> \models \psi$ is not captured in 1st-order terms using the language $\mathcal{L}$, but rather in 2nd-order terms, where also quantification over *subsets*, rather than only over *elements* of $\mathbf{R}$ is used:

$\forall A, B \subseteq \mathbf{R} ((A \ll B) \rightarrow \exists c \in \mathbf{R} (A \leq \{c\} \leq B))$,
where $A \ll B$ means that $a < b$ for every $a \in A$ and every $b \in B$ etc.