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B1.1 Logic

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Slides by **J. Koenigsmann** with some small additions; further reference see: **D. Goldrei**, “Propositional and Predicate Calculus: A Model of Argument”, Springer.

Introduction

1. What is mathematical logic about?

- provide a uniform, unambiguous **language** for mathematics

- make precise what a **proof** is

- explain and guarantee **exactness, rigor and certainty** in mathematics

- establish the **foundations** of mathematics

B1 (Foundations)
= B1.1 (Logic) + B1.2 (Set theory)

N.B.: Course does not teach you to think logically, but it explores what it *means* to think logically

2. Historical motivation

- *19th cent.*:
  need for conceptual foundation in analysis:
  what is the correct notion of
  infinity, infinitesimal, limit, ...

- attempts to formalize mathematics:
  - *Frege*'s Begriffsschrift
  - *Cantor*'s **naive** set theory:
  a set is any collection of objects

- led to **Russell's paradox**:
  consider the set $R := \{S \text{ set } \mid S \notin S\}$

$$R \in R \;\Rightarrow\; R \notin R \quad \text{contradiction}$$
$$R \notin R \;\Rightarrow\; R \in R \quad \text{contradiction}$$

$\leadsto$ *fundamental crisis in the foundations of mathematics*

3. Hilbert's Program

1. find a uniform (formal) **language** for all mathematics
2. find a complete system of **inference rules/ deduction rules**
3. find a complete system of mathematical **axioms**
4. prove that the system 1.+2.+3. is **consistent**, i.e. does not lead to contradictions

- ⋆ **complete:** every mathematical sentence can be proved or disproved using 2. and 3.
- ⋆ 1., 2. and 3. should be **finitary/effective/computable/algorithmic** so, e.g., in 3. you can't take as axioms *the system of all true sentences in mathematics*
- ⋆ **idea:** any piece of information is of finte length

4. Solutions to Hilbert's program

Step 1. is possible in the framework of
ZF = *Zermelo-Fraenkel set theory* or
ZFC = **ZF** + *Axiom of* **C***hoice*
(this is an empirical fact)
$\rightsquigarrow$ B1.2 Set Theory HT 2017

Step 2. is possible in the framework of
1st-order logic:
Gödel's Completeness Theorem
$\rightsquigarrow$ B1.1 Logic - this course

Step 3. is not possible ($\rightsquigarrow$ C1.2):
Gödel's 1st Incompleteness Theorem:
there is no effective axiomatization
of arithmetic

Step 4. is not possible ($\rightsquigarrow$ C1.2):
Gödel's 2nd Incompleteness Theorem, (but..)

5. Decidability

Step 3. of Hilbert's program fails:
there is no effective axiomatization
for the entire body of mathematics

But: many important parts of mathematics are completely and effectively axiomatizable, they are **decidable**, i.e. there is an *algorithm = program = effective procedure* deciding whether a sentence is true or false
$\rightsquigarrow$ allows proofs by computer

Example: $Th(\mathbf{C})$ = the **1st-order theory** of $\mathbf{C}$
= all *algebraic* properties of $\mathbf{C}$:

Axioms = *field axioms*
+ *all non-constant polynomials have a zero*
+ *the characteristic is* 0

Every algebraic property of $\mathbf{C}$ follows from these axioms.
Similarly for $Th(\mathbf{R})$.
$\rightsquigarrow$ C1.1 Model Theory

6. Why *mathematical* logic?

1. Language and deduction rules are tailored for *mathematical objects* and mathematical ways of reasoning

   N.B.: Logic tells you what a proof *is*, not how to *find* one

2. The *method* is mathematical:
   we will develop logic as a *calculus* with sentences and formulas
   $\Rightarrow$ Logic is itself a mathematical discipline,
   not meta-mathematics or philosophy,
   no ontological questions like
   what is a number?

3. Logic has *applications* towards other areas of mathematics, e.g. Algebra, Topology, but also towards theoretical computer science