2 Simple birth processes and continuous-time Markov chains

Questions 1-2 are starter questions and you can refer to them if you struggle with the harder questions. Questions 1-2 as well as 7-10 will not be marked.

1. [Not to be marked] Explain the evolution of a continuous-time Markov chain with Q matrix

$$Q = \begin{pmatrix} -4 & 2 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 3 & 0 & -5 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

In particular, write down holding rates and transition probabilities of the jump chain. Draw a diagram to represent the chain.

2. [Not to be marked] Suppose that we have a Q-matrix Q on a finite state space. Show that 0 is always an eigenvalue of Q.

3. Each bacterium in a colony splits into two identical bacteria after an exponential time of parameter λ, which then split independently in the same way. Let X_t denote the size of the colony at time t, and suppose $X_0 = 1$.

   (a) Show that $(X_t)_{t\geq 0}$ is a $(1, (\lambda_n)_{n\geq 0})$-birth process. What are its birth rates $(\lambda_n)_{n\geq 1}$?

   (b) Condition on the first splitting time to show that the probability generating function $\phi(t) = \mathbb{E}(z^{X_t})$, for any $z \in (-1, 1)$, satisfies

$$\phi(t) = ze^{-\lambda t} + \int_0^t \lambda e^{-\lambda s}\phi(t-s)^2 ds$$

   Show that this implies $\phi'(t) = \lambda\phi(t)(\phi(t) - 1)$.

   (c) Calculate the probability generating function of the geometric distribution with probability function $p_n = q^{n-1}(1-q)$, $n \geq 1$, geom$(1-q)$ for short. Deduce that, for $q = 1 - e^{-\lambda t}$ and $n = 1, 2, \ldots$

$$\mathbb{P}(X_t = n) = q^{n-1}(1-q).$$

   (d) Let $G_1, G_2, \ldots, G_m \sim$ geom$(1-q)$ be independent. Find the probability function of $\sum_{i=1}^m G_i$.
   Hint: this sum has the same distribution as the number of trials up to and including the mth success in a sequence of Bernoulli trials with probability $1-q$ *of success. This distribution is known as the* negative binomial *distribution.*

   (e) Show that

$$\mathbb{P}(X_u = k \,|\, X_0 = m) = \binom{k-1}{m-1}(1-e^{-\lambda u})^{k-m}(e^{-\lambda u})^m$$

   for $u \geq 0$ and $1 \leq m \leq k$.

   (f) Let $s \leq t$. Use the Markov property at s to calculate $\mathbb{P}(X_s = m, X_t = k)$ and hence $\mathbb{P}(X_s = m \,|\, X_t = k)$ for all $1 \leq m \leq k$.

4. Simple birth-and-death process. Consider a population in which each individual gives birth after an Exp(λ) amount of time, independently and repeatedly, and has a lifetime which is distributed as Exp(μ), independently of the births and of the other individuals.

(a) Explain why the number X_t of individuals alive at time t, evolves as a continuous-time Markov chain. What are its transition rates $Q_{i,j}, i, j \in \mathbb{N}$?

(b) Draw a diagram of the states of X, and draw a typical path, i.e. plot X_t against t.

(c) What is the transition matrix of the associated jump chain?

(d) Show that X does not explode.

Hint: Compare the process to a suitable simple birth process.

5. Let X be the same simple birth-death process as in the previous question (i.e. individuals have independent $\text{Exp}(\mu)$ lifetimes and, during their lifetime, give birth at rate λ repeatedly and independently of other individuals). It is clearly possible that the population dies out. Let

$$T = \inf\{t \geq 0 \colon X_t = 0\}$$

be the *extinction time* for the population.

(a) Write down the forward equations for this chain.

(b) Suppose now that $X_0 = 1$. Let $G(s,t) = \mathbb{E}(s^{X_t})$ be the probability generating function of X_t. Show that G satisfies

$$\frac{\partial}{\partial t} G(s,t) = (\lambda s - \mu)(s-1) \frac{\partial}{\partial s} G(s,t).$$

(c) It can be shown that the solution to this equation is

$$G(s,t) = \begin{cases} \dfrac{\mu(s-1) - (\lambda s - \mu)e^{-(\lambda-\mu)t}}{\lambda(s-1) - (\lambda s - \mu)e^{-(\lambda-\mu)t}} & \text{if } \lambda \neq \mu \\ \dfrac{\lambda t(s-1) - s}{\lambda t(s-1) - 1} & \text{if } \lambda = \mu. \end{cases}$$

Using the fact that $G(0,t) = \mathbb{P}(X_t = 0)$, find the distribution function and density of T. Find $\mathbb{E}(T)$ in the case $\lambda \leq \mu$. *Hint:* $\mathbb{E}(T) = \int_0^\infty \mathbb{P}(T > t)dt$.

(d) Using the expressions for $\mathbb{E}(s^{X_t})$ and the continuity theorem for probability generating functions, identify the limiting distribution of X_t as $t \to \infty$.

Hint: $\mathbb{P}(X_\infty \in \{0, \infty\}) = 1$.

6. **Two-state Markov chain.** Consider the Markov chain with Q-matrix $Q = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}$.

(a) Write down the backward and forward equations. Solve either the backward or the forward equations for the transition probabilities $p_{ij}(t)$, $i, j = 1, 2$. Check that your solution also satisfies the other equations.

Hint: one set is easier to solve than the other. The solutions of $y' = a + by$ *are* $y(x) = ce^{bx} - a/b$, $c \in \mathbb{R}$.

(b) Solve the equation $\xi Q = 0$ for ξ and verify that $p_{ij}(t) \to \xi_j$ as $t \to \infty$. This proves convergence to equilibrium by bare-hands methods, not requiring the general theory presented in the lectures.

The following questions are meant to deepen your understanding of the earlier material and/or go a little beyond the scope of the course. There will probably not be time for them to be covered in the classes and they will not be marked, but full solutions will be given on the solution sheets.

7. Let (U, V, W) be jointly continuously distributed random variables with joint probability density function $f_{U,V,W}$ that is continuous on $[0, \infty)^3$.

(a) Show that

$$\begin{aligned}\mathbb{P}(U \le a, W \le c \,|\, V = v) &:= \int_0^a \int_0^c f_{U,W|V=v}(u,w)\,dw\,du \\ &= \lim_{\varepsilon \downarrow 0} \mathbb{P}(U \le a, W \le c \,|\, v \le V \le v + \varepsilon).\end{aligned}$$

(b) Show that $\mathbb{P}(U \le a, W \le c \,|\, V = v) = \mathbb{P}(U \le a \,|\, V = v)\mathbb{P}(W \le c \,|\, V = v, U \le a)$.

(c) Show that $\mathbb{P}(W \le c \,|\, V = v, U \le a) = \mathbb{P}(W \le c)$ if (U, V) and W are independent.

(d) State generalisations of (a), (b) and (c) to the case when $U = (U_1, \ldots, U_k)$, $V = (V_1, \ldots, V_m)$ and $W = (W_1, \ldots, W_n)$. Comment briefly on changes needed to adapt the proofs of (a), (b) and (c).

8. **Complete proof of the Markov property for simple birth processes.** Let $(X_t)_{t\ge 0}$ be a $(k, (\lambda_n)_{n\ge 0})$-birth process with birth times $(T_n)_{n\ge 1}$. Set $Z_n := T_{n+1} - T_n$, $n \ge 0$.

(a) Let $t > 0$. Show that

$$\begin{aligned}&\mathbb{P}(Z_0 \le a, Z_1 \le b, T_2 \le t < T_2 + Z_2, Z_2 \le t - T_2 + c) \\ &\qquad = \mathbb{P}(Z_0 \le a, Z_1 \le b, T_2 \le t < T_2 + Z_2)\mathbb{P}(Z_2 \le c).\end{aligned}$$

Hint: condition on T_2 and apply results from Question 6.

(b) Let $t > 0$ and $\ell \ge k$. Explain briefly why

$$\begin{aligned}&\mathbb{P}(Z_0 \le a_0, \ldots, Z_{\ell-k-1} \le a_{\ell-k-1}, T_{\ell-k} \le t < T_{\ell-k} + Z_{\ell-k}, \\ &\qquad Z_{\ell-k} \le t - T_{\ell-k} + c_0, Z_{\ell-k+1} \le c_1, \ldots, Z_{\ell-k+m} \le c_m) \\ &= \mathbb{P}(Z_0 \le a_0, \ldots, Z_{\ell-k-1} \le a_{\ell-k-1}, T_{\ell-k} \le t < T_{\ell-k} + Z_{\ell-k}) \\ &\qquad \mathbb{P}(Z_{\ell-k} \le c_0, \ldots, Z_{\ell-k+m} \le c_m).\end{aligned}$$

(c) Deduce the Markov property for $(X_t)_{t\ge 0}$, i.e. that for all $t \ge 0$ and $\ell \ge k$, the pre-t process $(X_r)_{r\le t}$ and the post-t process $(X_{t+s})_{s\ge 0}$ are conditionally independent given $X_t = \ell$, and that the post-t process is an $(\ell, (\lambda_n)_{n\ge 0})$-birth process.

9. Let $(X_t)_{t\ge 0}$ be a Poisson process with jump times $(T_n)_{n\ge 1}$, and $L \ge 0$ an independent random variable. Using the Markov property, or otherwise, show the independence of

(a) $(X_r)_{r\le t}$ and $(X_{t+s} - X_t)_{s\ge 0}$,

(b) $(X_r)_{r\le T}$ and $(X_{T+s} - X_T)_{s\ge 0}$, where (i) $T = L$, (ii) $T = T_n$, or (iii) $T = \min\{T_n, L\}$.

10. Consider a hermaphrodite population (i.e. individuals are both male and female as, for example, with worms). Any individual can mate with any other, and a specific pair of individuals have a (single) child at rate λ. Suppose that initially, there are two individuals and let X_t denote the number at time t. Show carefully that $(X_t)_{t\ge 0}$ is a simple birth process. Calculate its birth rates $(\lambda_n)_{n\ge 2}$. Let $T_1, T_2, \ldots$ be the jump times of the birth process and let $T_\infty = \lim_{n\to\infty} T_n$. Show that X is explosive and calculate $\mathbb{E}(T_\infty)$.

Hint: $\frac{1}{k} - \frac{1}{k+1} = \frac{1}{k(k+1)}$.