3 Sheet 3: Continuous-time Markov chains

Questions 7 to 13 will not be marked. This assignment sheet is due for your 3rd class.

1. Consider a continuous-time Markov chain X on $\mathbb{S} = \mathbb{N}$ with Q-matrix Q, where the only non-zero off-diagonal entries of Q are

$$q_{n,n+1} = 2^n, \quad n \geq 0, \qquad q_{n,n-1} = 2^n, \quad n \geq 1.$$

   (a) Determine the transition probabilities of the underlying jump chain M and show that M is null recurrent.

   (b) Show that X has a unique stationary distribution and deduce that X is positive recurrent.

   (c) (optional) Find a null recurrent continuous-time Markov chain whose underlying jump chain M is positive recurrent.

2. Consider Markov chains X and Y with Q-matrices on $\{1, 2, 3, 4\}$ and $\{1, 2, 3, 4, 5\}$

$$Q_X = \begin{pmatrix} -1 & 1/2 & 1/2 & 0 \\ 1/4 & -1/2 & 0 & 1/4 \\ 1/6 & 0 & -1/3 & 1/6 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Q_Y = \begin{pmatrix} -3 & 2 & 0 & 0 & 1 \\ 0 & -3 & 3 & 0 & 0 \\ 0 & 5 & -5 & 0 & 0 \\ 0 & 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

   (a) What are the communicating classes. For each class, say if it is open or closed, and recurrent or transient.

   (b) For X, calculate the expected time to hit 4 starting from 1

   (c) For X, calculate the probability of hitting 3 starting from 1.

   (d) For Y, determine all stationary distributions.

   (e) (optional) For Y, determine the limit distribution when starting from 1.

   Hint: For (b) *and* (c) *you may wish to consider the quantities for arbitrary starting points and derive linear equations by conditioning on the first transition (time and state).*

3. Consider the M/M/1 queue, that is a single-server queue in which customers arrive in a Poisson process of rate λ and service times are independent identically exponentially distributed with parameter μ. Let X_t denote the length of the queue at time t including any customer being served, where $X_0 = 0$.

   (a) If the queue length is $k \geq 1$, what is the probability that the next customer arrives before the current customer's service time ends?

   (b) Describe the 'jump chain' M of X.

   (c) Determine the distribution of the number of arrivals during an Exp(μ) service period.

4. Let X be the length of an M/M/1 queue, as in the previous question, but now assume that $\lambda < \mu$.

   (a) Find the invariant distribution of X

   (b) Find the invariant distribution of the jump chain M.

   (c) Formulate the ergodic theorems for X and M. Use this to explain why the invariant distributions in (a) and (b) are different.

5. Let X be a renewal process whose inter-renewal times $(Z_n)_{n\geq 0}$ satisfy $0<\sigma^2=Var(Z_1)<\infty$ and $\mu=\mathbb{E}(Z_1)$. Deduce from the Central Limit Theorem for $(Z_n)_{n\geq 0}$ that

$$\frac{X_t - t/\mu}{\sqrt{t\sigma^2/\mu^3}} \to Z \sim \text{Normal}(0,1) \qquad \text{in distribution, as } t\to\infty.$$

Hint: Express probabilities involving X_t in terms of T_n.

6. **Proof of the Ergodic Theorem.** Let X be an irreducible positive recurrent continuous-time Markov chain on a countable state space $\mathbb{S}$, with holding time parameters λ_i and mean passage times m_i, $i\in\mathbb{S}$. Denote by $H_i^{(m)}$, $m\geq 1$, the successive passage times of X in i.

   (a) Fix $i\in\mathbb{S}$ and let $X_0=i$. Show that the increments $Z_m=S_{m+1}-S_m$ of $S_m=H_i^{(m)}$, $m\geq 0$, form a sequence of independent and identically distributed random variables. *Hint: Use the strong Markov property at S_m, $m\geq 1$.*

   (b) Fix $i\in\mathbb{S}$, as in (a). Let $X_0=i$. Show that

   $$\frac{S_m}{m}\to m_i=\mathbb{E}(Z_1) \qquad \text{almost surely, as } m\to\infty.$$

   What if $X_0=j$ for some $j\in\mathbb{S}$ with $j\neq i$?
   Hint: Only the distribution of Z_0 is different now. Consider Z_0 separately.

   (c) Prove the following form of the ergodic theorem.

   $$\frac{1}{t}\int_0^t 1_{\{X_s=i\}}ds \to \frac{1}{m_i\lambda_i} \qquad \text{almost surely, as } t\to\infty.$$

   Hints: Use (b) and also apply the strong law of large numbers to the holding times at i. Consider $t=H_i^{(m)}$, $m\to\infty$, first and deduce the general statement.

The following questions are meant to deepen your understanding of the earlier material and/or go a little beyond the scope of the course. There will probably not be time for them to be covered in the classes and they will not be marked, but full solutions will be given on the solution sheets.

7. (a) **Detailed balance equations.** Let $(X_t)_{t\geq 0}$ be a continuous-time Markov chain with Q-matrix $Q=(q_{ij})_{i,j\in\mathbb{S}}$. Suppose that a distribution $\xi=(\xi_i)_{i\in\mathbb{S}}$ satisfies

   $$\xi_i q_{ij}=\xi_j q_{ji}, \qquad \text{for all } i,j\in\mathbb{S} \qquad \text{(detailed balance equations).}$$

   Show that this implies that $\xi Q=0$, i.e. that ξ is an invariant distribution of $(X_t)_{t\geq 0}$.

   (b) Consider the Q-matrix $Q=\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$. What does this example tell you about the converse of (a)?

8. For $n\geq 1$, let $X_n=\begin{cases} n^3 & \text{with probability } n^{-2} \\ 0 & \text{with probability } 1-n^{-2} \end{cases}$

   (a) Show that $X_n\to 0$ in probability, as $n\to\infty$, but that $\mathbb{E}(X_n)\not\to 0$, as $n\to\infty$.

   (b) (optional) Show that $X_n\to 0$ almost surely.

   Now suppose that $(S_n)_{n\geq 0}$ is a simple symmetric random walk on $\mathbb{Z}$, started from $S_0=0$. Let $B_n=1_{\{S_n=0\}}$, so $B_n=1$ if $S_n=0$ and $B_n=0$ otherwise.

   (c) Show that $B_n\to 0$ in probability, as $n\to\infty$.

(d) We know that $(S_n)_{n\geq 0}$ is recurrent. Use this fact to show that B_n does not converge almost surely, as $n \to \infty$.

9. Customers arrive at a store as a Poisson process of rate 2. At the door, two representatives separately demonstrate the same product to anybody entering the store. Each demonstration takes a time which is exponentially distributed with parameter 1, and is independent of other demonstrations. After the demonstration the customer enters the store. When both representatives are busy, customers go directly into the store. If both representatives are free at $t = 0$, show that the probability that both are busy at $t > 0$ is

$$\frac{2}{5} - \frac{2}{3}e^{-2t} + \frac{4}{15}e^{-5t}.$$

Hint: You do not want to count customers in *the shop. What is your Markov chain? Show that the process described is indeed a continuous-time Markov chain.*

10. **Strong Markov property.** Let X be a continuous-time Markov chain started from some state $k \in \mathbb{S}$.

    (a) Show that the random times defined inductively by $T_k^0 = 0$ and, for $n \geq 0$,

    $$S_k^n := \inf\{t > T_k^n \colon X_t \neq k\}, \qquad T_k^{n+1} := \inf\{t > S_k^n \colon X_t = k\}$$

    are stopping times, i.e. the events $\{S_k^n \leq t\}$ and $\{T_k^n \leq t\}$ can be expressed in terms of $(X_s, s \leq t)$, for all $t \geq 0$ and $n \geq 0$.

    Remark: $T_k^1, T_k^2, \ldots$ are the successive return times to k.

    (b) Suppose now that $\mathbb{P}(T_k^1 < \infty) = 1$. What can you say using the Strong Markov Property about the sequence of random variables $(T_k^{n+1} - S_k^n)_{n\geq 0}$?

11. Let X be a continuous-time Markov chain in a finite state space $\mathbb{S}$, with Q-matrix Q and transition matrices $P(t)$, $t \geq 0$.

    (a) **Infinitesimal definition of Markov chains.** Show that for all $i, j \in \mathbb{S}$, as $h \downarrow 0$

    $$p_{ij}(h) = \delta_{ij} + q_{ij}h + o(h) \tag{1}$$

    where δ_{ij} is defined as 1 if $i = j$ and 0 otherwise.

    Hint: Derive lower bounds by restricting to the event that only one jump occurs before time h. Deduce upper bounds by summing over $j \in \mathbb{S}$.

    Remark: It can be shown that a right-continuous process $(X_t)_{t\geq 0}$ on $\mathbb{S}$ is a continuous-time Markov chain if and only if X_{t+h} is conditionally independent of $(X_r)_{r\leq t}$ given $X_t = i$ (Markov property) and $P(X_{t+h} = j | X_t = i) = \delta_{ij} + q_{ij}h + o(h)$ uniformly in t.

    (b) **A proof that forward equations hold.** Show that, as $h \downarrow 0$,

    $$\frac{p_{ik}(t+h) - p_{ik}(t)}{h} = \sum_{j\in\mathbb{S}} p_{ij}(t)q_{jk} + o(1)$$

    and deduce that $P(t)$, $t \geq 0$, satisfies the forward equation $P'(t) = P(t)Q$, $P(0) = I$.

12. **Matrix exponentials.** Consider a Q-matrix Q on a finite state space.

    (a) Show that $e^{tQ} := \sum_{n\geq 0} \frac{t^n}{n!} Q^n$, with Q^0 the identity matrix, converges componentwise.

(b) Consider componentwise differentiation $\frac{d}{dt}e^{tQ}$. Show that $\frac{d}{dt}e^{tQ} = Qe^{tQ} = e^{tQ}Q$. Deduce that $A(t) := e^{tQ}$, $t \geq 0$, are the transition matrices of a Q-Markov chain.

Hint: Use the formula for products of power series to see that $A(s)A(t) = A(s+t)$, and show that $A(t)$ has non-negative entries first for the case $q_{ij} > 0$ for all $i \neq j$.

(c) For the two-state Markov chain of Question 4, calculate Q^n by setting up recurrence relations for its entries (or otherwise) and hence find e^{tQ}. Compare your answer with Question 4(a).

13. **A non-minimal continuous-time Markov chain.** Consider the explosive birth process with rates $\lambda_n = 2^n$, $n \geq 0$. Denote its Q-matrix by Q. Let $Z_n^{(m)} \sim \text{Exp}(\lambda_n)$, $n \geq 0$, $m \geq 1$, be independent. Let $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ and define a family of times $(T_n^{(m)})_{n \in \overline{\mathbb{N}}, m \geq 1}$ by

$$T_0^{(1)} = 0, \qquad T_n^{(m)} = T_0^{(m)} + \sum_{k=0}^{n-1} Z_k^{(m)}, \quad n \in \overline{\mathbb{N}}, \quad T_0^{(m+1)} = T_\infty^{(m)}, \qquad m \geq 1.$$

Define a process X taking values in $\mathbb{N}$ by

$$X_t = n \qquad \text{if } T_n^{(m)} \leq t < T_{n+1}^{(m)}, \qquad n \geq 0, m \geq 1.$$

Note that this just means that the birth process starts afresh from 0 after explosion. You may assume that this defines X_t for all $t \in [0, \infty)$, with probability 1. (This can be proved using the Strong Law of Large Numbers, see later in the course). We will also talk about the process started from points other than 0; by the process started at $i \in \mathbb{N}$, we mean the process $(X_{T_i^{(1)}} + t)_{t \geq 0}$ (conditionally given $T_i^{(1)} < \infty$, which occurs with probability 1). In any case, we will denote the fist jump time of X by J_1.

(a) Calculate all non-negative solutions ξ of $\xi Q = 0$.

(b) For $i \in \mathbb{N}$, let $H_i^{(1)} = H_i = \inf\{t > J_1 : X_t = i\}$ denote the first passage time to i. Calculate $m_i = \mathbb{E}_i(H_i)$ and define $\xi_i = 1/(m_i \lambda_i)$. Note that ξ is a probability distribution. What would ξ represent in the non-explosive case and what is different here?

Hint: what is the relationship between $T_\infty^{(1)}$ and H_0?

(c) Prove the following as far as possible. Given $X_0 = i$, the passage time H_j to $j > i$, has density function

$$f_{ij}^{(1)} = f_{ij} = g_{\lambda_i} * \cdots * g_{\lambda_{j-1}},$$

where g_λ denotes the density of $\text{Exp}(\lambda)$ and $*$ denotes convolution $(g * h)(x) = \int_0^x g(y)h(x-y)dy$. Furthermore, if $j > i$ then, given $X_0 = j$, the density of H_i is a quantity f_{ji} satisfying $f_{ij} * f_{ji} = f$, where $f = f_{ii} = f_{00}$ is the density of T_∞ (which you may assume to exist, as well as f_{ji}). The densities of successive passage times $H_j^{(m+1)}$, $m \geq 1$, are then $f_{ij}^{(m+1)} = f_{ij} * f^{*(m)}$ where $\cdot^{*(m)}$ denotes mth convolution power, i.e. $f^{*(1)} = f$ and $f^{*(m+1)} = f * f^{*(m)}$.

Hint: remember that the density of the sum of continuous random variables is the convolution product of their densities.

(d) Show that for all $t \geq 0$ and $i, j \in \mathbb{N}$

$$p_{ij}(t) := \mathbb{P}_i(X_t = j) = \frac{1}{\lambda_j} \sum_{m \geq 1} f_{i,j+1}^{(m)}(t).$$

(e) Show that $\xi P(t) = \xi$ for all $t \geq 0$.