B8.3 — possible topics for MMSC MSc projects

Here is a brief description of some possible financial mathematics topics for students on the Mathematical Modelling and Scientific Computing MSc.

More complex models

The Black-Scholes model results in some predictions which are at variance with real world observations. The most important of these is as follows. When we write

$$\frac{dS_t}{S_t} = (\mu - y)\,dt + \sigma\,dW_t$$

we are saying the volatility, σ, is a property of the underlying asset rather than option. If we look at the Black-Scholes formula for a call option's price

$$C(S,t;K,T,r,y,\sigma) = S\,e^{-y(T-t)}\,\mathrm{N}(d_+) - K\,e^{-r(T-t)}\,\mathrm{N}(d_-),$$

$$d_\pm = \frac{\log(S/K) + (r - y \pm \frac{1}{2}\sigma^2)(T-t)}{\sqrt{\sigma^2(T-t)}}$$

we see that the only parameter which is not directly observable is σ, the volatility—the share price S is quoted on an exchange, K and T are written into the option contract, t is the current time, r is quoted by banks, y can be computed from historical data and the option's price C_m is quoted on option exchanges. Given that a call's vega, $\partial C/\partial\sigma$, is strictly positive it follows that if we know S, t, K, T, r, y and C_m, the market price of the option, then we can compute the implied volatility, σ_{imp}, by solving (numerically) the equation

$$C_m = C(S,t;K,T,r,y,\sigma_{\mathrm{imp}}).$$

As volatility is supposed to be a property of the underlying asset, if at any moment we compute the implied volatility for call options on the same underlying asset but with a range of different strikes we should get the same implied volatility. In practice we don't. This observation has lead people to consider a variety of other, more complicated models. Two of the most important are as follows.

Local volatility surfaces

In this model it is assumed that volatility is not a constant, rather it is a function of S and t,

$$\frac{dS_t}{S_t} = (\mu - y)\,dt + \sigma(S_t,t)\,dW_t.$$

This leads to the pricing equation

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma(S,t)^2\,S^2\,\frac{\partial^2 V}{\partial S^2} + (r-y)\,S\,\frac{\partial V}{\partial S} - r\,V = 0.$$

It is possible (by playing around with forward and backward Kolmogorov equations—the Feynman Kăc equation is the same things as the backward Kolmogorov equation) to extract the function $\sigma(S,t)$ from the market prices of calls across a range of strikes and expiry dates. The most common technique is that of Bruno Dupire. A mini-project in this area might look at Dupire's method and implement it on some test, or real-world, data.

Stochastic volatility models

In this view of the world, the volatility itself follows a SDE and the model takes the form

$$\begin{aligned} \frac{dS_t}{S_t} &= (\mu - y)\,dt + \sigma_t\,dW_t, \\ d\sigma_t &= \alpha(\sigma_t,t)\,dt + \beta(\sigma_t,t)\,dW_t, \end{aligned}$$

for suitable functions $\alpha(\sigma,t)$ and $\beta(\sigma,t)$. There are a number of directions a project in this area could take.

1. The formulation and numerical solution of Heston's stochastic volatility model. Heston chose the functions α and β in such a way that the pricing equation could be solved exactly (for a call) in the sense that an analytic expression for the Fourier transform of the call's price can be found. It is not possible to find an analytic expression for the call's price itself, and the Fourier transform has to be inverted numerically.

2. The stochastic volatility models of Papanicoulou *et al.* These are models set up so that it is possible to find asymptotic expansions of an option's price in terms of a small parameter (arising in the SDE for the volatility) in such a way that (to the first few orders) the only quantity that needs to be measured is the Black-Scholes implied volatility—rather than the instantaneous volatility σ_t. This is a great advantage as it is not clear how one could measure σ_t. A project in this area would involve reading one of the papers by Papanicoulou *et al.* and reporting on the details.

3. The SABR model of Hagan *et al.* This is a widely used model (in the interest rate world) which is also based on an asymptotic expansion of a stochastic volatility model and is used to price interest rate caps (effectively, call options on an interest rate). A project in this are would involve reporting on the details of one of the papers by Hagan *et al.*

4. The VIX index is a measure of the variance of the S&P-500 index devised by the CBOE (and VIX index options and forwards are traded on the CBOE). This model makes no assumptions about the SDE that the volatility follows, but rather devises a scheme by which it

can be measured directly from the market prices of puts and calls. A project in this area would involve a description, derivation and possible implementation of this scheme.

Numerical methods

There are many option pricing problems which do not have analytic solutions and must be solved numerically. The most common numerical methods used are:

- finite-difference methods;
- binomial methods;
- Monte-Carlo methods; and, to a lesser degree,
- Fourier methods.

Each method has its strength and weaknesses; some work better than other for certain classes of options. Possible projects in this area might include:

- Comparison of the methods for various types of exotic options (Asians which depend on the average asset price, lookbacks which depend on the maximum or minimum of the asset price, forward starts where the strike is determined by the value of the asset at some future time, etc.)
- Monte Carlo techniques for computing the greeks, such as Δ and Γ.
- Finite difference and/or binomial methods for American options. The binomial method for an American option is fairly straight-forward. The American option problem can cast as what is known as a linear complementarity formulation which is particularly suitable for finite-difference methods (specifically, iterative methods known as projection methods which are very similar to Gauss-Seidel methods for linear systems and penalty methods which are very similar to Newton's method for solving non-linear systems).

Asymptotic methods

Frequently one of the parameters r, y, σ or $T - t$ is small compared to the others. By making the financial variables dimensionless in the appropriate way this leads to a small parameter in the dimensionless Black-Scholes equation, which can be exploited to obtain an asymptotic series solution.

When r or y is small, the resulting equation can be dealt with using regular asymptotic expansions (i.e., a power series in the small parameter).

When σ or $T - t$ is small, the resulting equation requires singular perturbation techniques (similar to boundary layer analysis in high Reynolds' number flows). The case of small time to expiry, $T - t \ll 1$, is particularly interesting as it leads to a rational approach to smoothing payoffs whose discontinuities or kinks cause problems when estimating Δ and Γ from numerical solutions.

One can also use the method of multiple scales to deal with events such as many discretely paid dividends yields (and show that one can legitimately approximate them by a continuous dividend yield, at leading order).