B8.3 Mathematical Models for Financial Derivatives

Hilary Term 2019

Problem Sheet Zero

1. The moment generating function of a (real valued) random variable X is defined by $m_X(t) = \mathbb{E}\left[\, e^{tX} \,\right]$, assuming this expectation exists. Suppose that X is a normally distributed random variable with zero mean and variance σ^2, so its probability density function is

$$p(x) = \frac{1}{\sqrt{2\pi\,\sigma^2}} e^{-x^2/2\sigma^2}.$$

Show that in this case $m_X(t) = e^{\sigma^2 t^2/2}$. Hence deduce that the odd moments of X are all zero, i.e., for $n = 0, 1, 2, \ldots$

$$\mathbb{E}\left[\, X^{2n+1} \,\right] = 0,$$

and the even moments are given by

$$\mathbb{E}\left[\, X^{2n} \,\right] = \frac{(2n)!}{2^n\, n!}\,\sigma^{2n} = 1 \cdot 3 \cdots (2n-3)(2n-1)\,\sigma^{2n},$$

for $n = 0, 1, 2, \ldots$. *In particular, note for future reference that*

$$\mathbb{E}\left[\, X^4 \,\right] = 3\,\sigma^4.$$

2. A die has six faces labelled 1 to 6 and when rolled the probability that any given face appears is $\frac{1}{6}$. Find

   (a) The expected value of the die;

   (b) The expected value of the die if only an odd number appears;

   (c) The expected value of the die if only an even number appears.

   Give the probability that an odd number appears and the probability that an even number appears. Show that the expected value of the die, with no conditions, is the same thing as

$$\begin{pmatrix} \text{probability} \\ \text{of rolling} \\ \text{an even} \\ \text{number} \end{pmatrix} \times \begin{pmatrix} \text{the expected} \\ \text{value if an} \\ \text{even number} \\ \text{is rolled} \end{pmatrix} + \begin{pmatrix} \text{probability} \\ \text{of rolling} \\ \text{an odd} \\ \text{number} \end{pmatrix} \times \begin{pmatrix} \text{the expected} \\ \text{value if an} \\ \text{odd number} \\ \text{is rolled} \end{pmatrix}.$$

   This is a particular case of a general law, called the *tower law* or the *law of iterated expectations.*

3. For $t > 0$, let

$$p(y; x, t) = \frac{1}{\sqrt{2\pi\, t}} e^{-(x-y)^2/2t}.$$

This can be interpreted as the probability density function for a normal random variable Y which has mean x and variance t. Show, by direct calculation, that $p(y; x, t)$ also satisfies the heat equation

$$\frac{\partial p}{\partial t} = \tfrac{1}{2} \frac{\partial^2 p}{\partial x^2}, \quad \text{for } t > 0,\ x \in \mathbb{R}.$$

Hence deduce that

$$u(x, t) = \mathbb{E}\big[\, f(y) \,\big] = \frac{1}{\sqrt{2\pi\, t}} \int_{-\infty}^{\infty} f(y) e^{-(x-y)^2/2t}\, dy$$

satisfies the heat equation

$$\frac{\partial u}{\partial t} = \tfrac{1}{2} \frac{\partial^2 u}{\partial x^2}, \quad \text{for } t > 0,\ x \in \mathbb{R},$$

provided the integral converges absolutely. [Hint: you can assume that the absolute convergence means you can swap the order of partial differentiation and integration.]

Assuming that the integral converges absolutely and f is continuous at all points in $\mathbb{R}$, show that

$$\lim_{t \to 0^+} u(x, t) = \lim_{t \to 0^+} \frac{1}{\sqrt{2\pi\, t}} \int_{-\infty}^{\infty} f(y) e^{-(x-y)^2/2t}\, dy = f(x)$$

for each $x \in \mathbb{R}$. [Hint: change variables to $s = (y-x)/\sqrt{t}$ and then you may assume that the absolute convergence allows you to interchange the order of limit and integration.]

4. Show that if $u(x, t)$ is a solution of the heat equation

$$\frac{\partial u}{\partial t} = \tfrac{1}{2} \frac{\partial^2 u}{\partial x^2},$$

for $t > 0$ and all $x \in \mathbb{R}$, then so too is $u(-x, t)$. Write down solutions u_1 and u_2 of the heat equation, in terms of u (and for $t > 0$ and $x \geq 0$), which satisfy

$$(a) \quad u_1(0, t) = 0, \qquad (b) \quad \frac{\partial u_2}{\partial x}(0, t) = 0.$$

5. The Black-Scholes equation (with no dividend yields) is

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r\, S \frac{\partial V}{\partial S} - r\, V = 0$$

for $t < T$ and $S > 0$, where r and $\sigma > 0$ are constants (the interest rate and volatility). Note that we solve this equation *backwards* in time, from T (the expiry time, in the future) to the present.

By assuming a separable solution of the form $V(S,t) = F(S)\,G(t)$, find solutions which satisfy the *terminal* condition $V(S,T) = S^m$, $S > 0$, where m is a constant.

[Hint: start with the terminal condition before worrying about the differential equation.]