B8.3 Mathematical Models for Financial Derivatives

Hilary Term 2019

Problem Sheet Zero

1. The moment generating function of a (real valued) random variable X is defined by $m_X(t) = \mathbb{E}\big[\, e^{tX} \,\big]$, assuming this expectation exists. Suppose that X is a normally distributed random variable with zero mean and variance σ^2, so its probability density function is

$$p_X(x) = \frac{1}{\sqrt{2\pi\,\sigma^2}} e^{-x^2/2\sigma^2}.$$

Show that in this case $m_X(t) = e^{\sigma^2 t^2/2}$.

We have

$$\begin{aligned}
m_X(t) &= \mathbb{E}\big[\, e^{tX} \,\big] \\
&= \frac{1}{\sqrt{2\pi\,\sigma^2}} \int_{-\infty}^{\infty} e^{tx}\, e^{-x^2/2\sigma^2}\, dx \\
&= \frac{1}{\sqrt{2\pi\,\sigma^2}} \int_{-\infty}^{\infty} e^{-(x^2 - 2t\sigma^2 x + t^2\sigma^4 - t^2\sigma^4)/2\sigma^2}\, dx \\
&= \frac{e^{\sigma^2 t^2/2}}{\sqrt{2\pi\,\sigma^2}} \int_{-\infty}^{\infty} e^{-(x - t\sigma^2)^2/2\sigma^2}\, dx \\
&= e^{\sigma^2 t^2/2}.
\end{aligned}$$

Hence deduce that the odd moments of X are all zero, i.e., for $n = 0, 1, 2, \ldots$

$$\mathbb{E}\big[\, X^{2n+1} \,\big] = 0,$$

and the even moments are given by

$$\mathbb{E}\big[\, X^{2n} \,\big] = \frac{(2n)!}{2^n\, n!}\,\sigma^{2n} = 1\cdot 3 \cdots (2n-3)(2n-1)\,\sigma^{2n},$$

for $n = 0, 1, 2, \ldots.$

There are various ways to do this. Mine is as follows. Write

$$e^{tX} = 1 + tX + \frac{t^2 X^2}{2!} + \cdots + \frac{t^{2n} X^{2n}}{(2n)!} + \frac{t^{2n+1} X^{2n+1}}{(2n+1)!} + \cdots,$$

so that

$$\begin{aligned}
&\mathbb{E}\big[e^{tX}\big] \\
&= 1 + t\mathbb{E}\big[X\big] + \frac{t^2}{2!}\mathbb{E}\big[X^2\big] + \cdots + \frac{t^{2n}}{(2n)!}\mathbb{E}\big[X^{2n}\big] + \frac{t^{2n+1}}{(2n+1)!}\mathbb{E}\big[X^{2n+1}\big] + \cdots.
\end{aligned}$$

Now expand $e^{\sigma^2 t^2/2}$ as a Taylor series about zero,

$$e^{\sigma^2 t^2/2} = 1 + \frac{\sigma^2 t^2}{2} + \frac{\sigma^4 t^4}{2^2\, 2!} + \frac{\sigma^6 t^6}{2^3\, 3!} + \cdots + \frac{\sigma^{2n} t^{2n}}{2^n\, n!} + \cdots ,$$

and then match the two series by powers of t. As the series for $e^{\sigma^2 t^2/2}$ has no odd powers of t, we must have

$$\frac{t^{2n+1}}{(2n+1)!} \mathbb{E}\big[X^{2n+1}\big] = 0,$$

which implies that

$$\mathbb{E}\big[X^{2n+1}\big] = 0,$$

for $n = 0, 1, 2, \ldots$.. Matching the even powers gives

$$\frac{t^{2n}}{(2n)!} \mathbb{E}\big[X^{2n}\big] = \frac{2^{-n} \sigma^{2n} t^{2n}}{n!}$$

and then cancelling the t^{2n} and rearranging gives

$$\mathbb{E}\big[X^{2n}\big] = \frac{(2n)!}{2^n\, n!}\, \sigma^{2n}.$$

In particular, note for future reference that

$$\mathbb{E}\big[\, X^4 \,\big] = 3\, \sigma^4.$$

Putting $n = 2$ here we get

$$\mathbb{E}\big[\, X^4 \,\big] = \frac{4!}{2^2\, 2!}\, \sigma^4 = \frac{4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 2} \sigma^4 = 3\, \sigma^4,$$

a quite famous result about the (dimensional) kurtosis of a normally distributed random variable. We use it in Week 3.

2. A die has six faces labelled 1 to 6 and when rolled the probability that any given face appears is $\frac{1}{6}$. Find

   (a) The expected value of the die;
   This is simply

   $$\tfrac{1}{6} \times 1 + \tfrac{1}{6} \times 2 + \tfrac{1}{6} \times 3 + \tfrac{1}{6} \times 4 + \tfrac{1}{6} \times 5 + \tfrac{1}{6} \times 6 = \tfrac{21}{6} = 3\,\tfrac{1}{2}.$$

   (b) The expected value of the die if only an odd number appears;
   There are three odd numbers which can appear and, given that an odd number does appear, they appear with equal probability of $\frac{1}{3}$. Therefore, given that an odd number does appear, the expected value is

   $$\tfrac{1}{3} \times 1 + \tfrac{1}{3} \times 3 + \tfrac{1}{3} \times 5 = \tfrac{9}{3} = 3.$$

(c) The expected value of the die if only an even number appears.

There are three even numbers which can appear and, given that an even number does appear, they appear with equal probability of $\frac{1}{3}$. Therefore, given that an even number does appear, the expected value is

$$\tfrac{1}{3} \times 2 + \tfrac{1}{3} \times 4 + \tfrac{1}{3} \times 6 = \tfrac{12}{3} = 4$$

Give the probability that an odd number appears and the probability that an even number appears.

Since all numbers are equally likely and there are three odd ones and three even ones,

$$\text{prob(even)} = \tfrac{3}{6} = \tfrac{1}{2}, \qquad \text{prob(odd)} = \tfrac{3}{6} = \tfrac{1}{2}.$$

Show that the expected value of the die, with no conditions, is the same thing as

$$\begin{pmatrix}\text{probability}\\ \text{of rolling}\\ \text{an even}\\ \text{number}\end{pmatrix} \times \begin{pmatrix}\text{the expected}\\ \text{value if an}\\ \text{even number}\\ \text{is rolled}\end{pmatrix} + \begin{pmatrix}\text{probability}\\ \text{of rolling}\\ \text{an odd}\\ \text{number}\end{pmatrix} \times \begin{pmatrix}\text{the expected}\\ \text{value if an}\\ \text{odd number}\\ \text{is rolled}\end{pmatrix}.$$

From above, the expected value given no conditions is $3\,\frac{1}{2}$. The expression immediately above evaluates to

$$\tfrac{1}{2} \times 3 + \tfrac{1}{2} \times 4 = \tfrac{7}{2} = 3\,\tfrac{1}{2}.$$

This is a particular case of a general law, called the *tower law* or the *law of iterated expectations.*

We use this idea in Week 4 and possibly in Week 5; it appears more complicated there, but it is really just the idea outlined above.

3. For $t > 0$, let

$$p(y; x, t) = \frac{1}{\sqrt{2\pi\, t}} e^{-(x-y)^2/2t}.$$

This can be interpreted as the probability density function for a normal random variable Y which has mean x and variance t. Show, by direct calculation, that $p(y; x, t)$ also satisfies the heat equation

$$\frac{\partial p}{\partial t} = \frac{1}{2} \frac{\partial^2 p}{\partial x^2}, \quad \text{for } t > 0,\ x \in \mathbb{R}.$$

Direct calculation gives

$$
\begin{aligned}
\frac{\partial p}{\partial x} &= -\frac{(x-y)}{t}\, p(y;x,t), \\
\frac{\partial^2 p}{\partial x^2} &= -\frac{1}{t}\, p(y;x,t) - \frac{(x-y)}{t}\,\frac{\partial p}{\partial x} \\
&= -\frac{1}{t}\, p(y;x,t) + \frac{(x-y)^2}{t^2}\, p(y;x,t), \\
\frac{\partial p}{\partial t} &= \frac{-1}{2\sqrt{2\pi\, t^3}}\, e^{-(x-y)^2/2t} + \frac{-(x-y)^2}{-2t^2}\, p(y;x,t) \\
&= \tfrac{1}{2}\left(-\frac{1}{t}\, p(y;x,t) + \frac{(x-y)^2}{t^2}\, p(y;x,t)\right) \\
&= \tfrac{1}{2}\,\frac{\partial^2 p}{\partial x^2}.
\end{aligned}
$$

Hence deduce that

$$u(x,t) = \mathbb{E}\big[\, f(y)\,\big] = \frac{1}{\sqrt{2\pi\, t}} \int_{-\infty}^{\infty} f(y) e^{-(x-y)^2/2t}\, dy$$

satisfies the heat equation

$$\frac{\partial u}{\partial t} = \tfrac{1}{2}\,\frac{\partial^2 u}{\partial x^2}, \quad \text{for } t > 0,\ x \in \mathbb{R},$$

provided the integral converges absolutely. [Hint: you can assume that the absolute convergence means you can swap the order of partial differentiation and integration.]

Write the solution in the form

$$u(x,t) = \int_{-\infty}^{\infty} f(y)\, p(y;x,t)\, dy$$

and assume that the integral is absolutely convergent, which means that

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial}{\partial t}\int_{-\infty}^{\infty} f(y)\, p(y;x,t)\, dy \\
&= \int_{-\infty}^{\infty} \frac{\partial}{\partial t}\left(f(y)\, p(y;x,t)\right) dy \\
&= \int_{-\infty}^{\infty} f(y)\, \frac{\partial p}{\partial t}(y;x,t)\, dy
\end{aligned}
$$

and, similarly,

$$\frac{\partial^2 u}{\partial x^2} = \int_{-\infty}^{\infty} f(y)\, \frac{\partial^2 p}{\partial x^2}(y;x,t)\, dy.$$

Thus

$$\frac{\partial u}{\partial t} - \tfrac{1}{2}\frac{\partial^2 u}{\partial x^2} = \int_{-\infty}^{\infty} f(y)\left(\frac{\partial p}{\partial t} - \tfrac{1}{2}\frac{\partial^2 p}{\partial x^2}\right) dy = 0$$

since

$$\frac{\partial p}{\partial t} - \tfrac{1}{2}\frac{\partial^2 p}{\partial x^2} = 0$$

identically for $t > 0$.

Assuming that the integral converges absolutely and f is continuous at all points in $\mathbb{R}$, show that

$$\lim_{t\to 0^+} u(x,t) = \lim_{t\to 0^+} \frac{1}{\sqrt{2\pi\, t}} \int_{-\infty}^{\infty} f(y) e^{-(x-y)^2/2t}\, dy = f(x)$$

for each $x \in \mathbb{R}$. [Hint: change variables to $s = (y-x)/\sqrt{t}$ and assume that the absolute convergence allows you to interchange the order of limit and integration.]

We have

$$\begin{aligned}
\lim_{t\to 0^+} u(x,t) &= \lim_{t\to 0^+} \frac{1}{\sqrt{2\pi\, t}} \int_{-\infty}^{\infty} f(y) e^{-(x-y)^2/2t}\, dy \\
&= \lim_{t\to 0^+} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f\left(x + s\sqrt{t}\right) e^{-s^2/2}\, ds \quad (s = (y-x)/\sqrt{t}) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \lim_{t\to 0^+} f\left(x + s\sqrt{t}\right) e^{-s^2/2}\, ds \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-s^2/2}\, ds \\
&= \frac{f(x)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-s^2/2}\, ds \\
&= f(x)
\end{aligned}$$

The reason we can write

$$\lim_{t\to 0^+} f\left(x + s\sqrt{t}\right) = f(x)$$

is that we assume that $f(y)$ is continuous for all $y \in \mathbb{R}$. If there is a point x where f is not continuous then, in general, the result

$$\lim_{t\to 0^+} u(x,t) = f(x)$$

does *not* apply at that point.

4. Show that if $u(x,t)$ is a solution of the heat equation

$$\frac{\partial u}{\partial t} = \tfrac{1}{2}\frac{\partial^2 u}{\partial x^2},$$

for $t > 0$ and all $x \in \mathbb{R}$, then so too is $u(-x,t)$.

Set $z = -x$ and look at $u(z,t) = u(-x,t)$. We have

$$\frac{\partial u}{\partial z}(z,t) = \frac{dz}{dx}\frac{\partial u}{\partial x}(z,t) = -\frac{\partial u}{\partial x}(z,t)$$

and, similarly,

$$\frac{\partial^2 u}{\partial z^2}(z,t) = \frac{\partial^2 u}{\partial x^2}(z,t).$$

Since the heat equation is satisfied for all $x \in \mathbb{R}$, it follows that

$$\frac{\partial u}{\partial t}(z,t) = \tfrac{1}{2}\frac{\partial^2 u}{\partial z^2}(z,t)$$

for all $t > 0$ and $z \in \mathbb{R}$. Now rename the variable z to x.

Write down solutions u_1 and u_2 of the heat equation, in terms of u (and for $t > 0$ and $x \geq 0$), which satisfy

$$(a) \quad u_1(0,t) = 0, \qquad (b) \quad \frac{\partial u_2}{\partial x}(0,t) = 0.$$

In case (a), try $u_1(x,t) = u(x,t) - u(-x,t)$. By linearity of the heat equation and the previous result, this is a solution of the heat equation. Then note that

$$u_1(0,t) = u(0,t) - u(0,t) = 0,$$

so the boundary condition is satisfied as well.

In case (b), try $u_2(x,t) = u(x,t) + u(-x,t)$. Again, by linearity of the heat equation it is clear that $u_2(x,t)$ is also a solution. This implies that $\partial^2 u_2/\partial x^2$ exists for all $x \in \mathbb{R}$, which in turn implies that $\partial u_2/\partial x$ exists for all $x \in \mathbb{R}$. Therefore we can write

$$\frac{\partial u_2}{\partial x}(0,t) = \lim_{\delta \to 0} \frac{u_2(\delta,t) - u_2(-\delta,t)}{2\,\delta},$$

but it is clear that

$$u_2(\delta,t) - u_2(-\delta,t) = \big(u(\delta,t) + u(-\delta,t)\big) - \big(u(-\delta,t) + u(\delta,t)\big) = 0$$

for any real δ, which implies that

$$\frac{\partial u_2}{\partial x}(0,t) = 0,$$

as required.

5. The Black-Scholes equation (with no dividend yields) is

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r\,S\,\frac{\partial V}{\partial S} - r\,V = 0$$

for $t < T$ and $S > 0$, where r and $\sigma > 0$ are constants (the interest rate and volatility). Note that we solve this equation *backwards* in time, from T (the expiry time, in the future) to the present.

By assuming a separable solution of the form $V(S,t) = F(S)\,G(t)$, find solutions which satisfy the *terminal* condition $V(S,T) = S^m$, $S > 0$, where m is a constant. [Hint: start with the terminal condition before worrying about the differential equation.]

As suggested, starting with the terminal condition gives

$$V(S,T) = F(S)\,G(T) = S^m$$

for all $S > 0$. The obvious solution is[1]

$$F(S) = S^m, \quad G(T) = 1.$$

Now observe that

$$\begin{aligned} \frac{\partial V}{\partial t} &= S^m\,\dot{G}(t), \\ S\,\frac{\partial V}{\partial S} &= m\,S^m\,G(t), \\ S^2\,\frac{\partial^2 V}{\partial S^2} &= m(m-1)\,S^m\,G(t), \end{aligned}$$

so the Black-Scholes equation reduces to

$$\Big(\dot{G}(t) + \tfrac{1}{2}\sigma^2\,m(m-1)\,G(t) + r\,m\,G(t) - r\,G(t)\Big)\,S^m = 0,$$

for all $S > 0$. This implies that $G(t)$ satisfies the ODE

$$\dot{G}(t) + \lambda\,G(t) = 0$$

where

$$\lambda = \left(\tfrac{1}{2}\sigma^2\,m + r\right)(m-1)\,.$$

Finally, applying the condition that $G(T) = 1$ we get

$$G(t) = e^{\lambda(T-t)}$$

[1] You could choose $F(S) = \alpha\,S^m$, $G(T) = 1/\alpha$ for any constant $\alpha \neq 0$, in which case you would end up with the same answer, but a more complicated calculation to get it.

and hence

$$V(S,t) = e^{\lambda(T-t)}\, S^m.$$

As an aside, this also gives us the two (linearly independent) steady-state solutions of the Black-Scholes equation (corresponding to $\lambda = 0$),

$$V_1(S) = S, \quad V_2(S) = S^{-2r/\sigma^2}$$

which occur when $m = 1$ and $m = -2r/\sigma^2$, respectively.