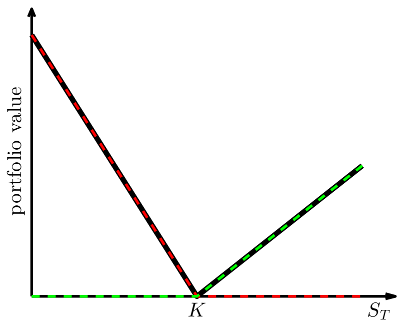
B8.3 Mathematical Models for Financial Derivatives

Hilary Term 2019

Problem Sheet One Solutions

1. Compute the payoffs and sketch the payoff diagrams for the following portfolios involving European options with expiry date T.

   (a) Long one call and two puts, all with strike K (this is a *strip*).



The payoff is

$$(S_T - K)^+ + 2\,(K - S_T)^+ = \begin{cases} 2\,(K - S_T) & \text{if } S_T \le K \\ S_T - K & \text{if } S_T > K \end{cases}$$

A speculator holding this portfolio would hope S_T would move away from K. If the move had absolute size $0 < \delta S \le K$ then the down move, $S_T = K - \delta S$, would lead to more profit than the up move, $S_T = K + \delta S$.

   (b) Long one put and two calls, all with strike K (this is a *strap*).

The payoff is

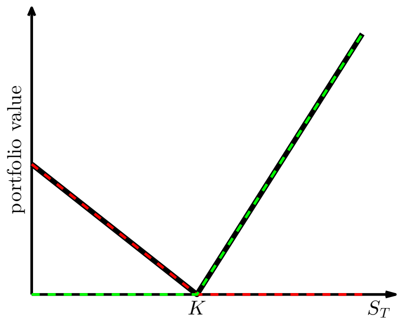
$$2\,(S_T - K)^+ + (K - S_T)^+ = \begin{cases} K - S_T & \text{if } S_T \le K \\ 2\,(S_T - K) & \text{if } S_T > K \end{cases}$$

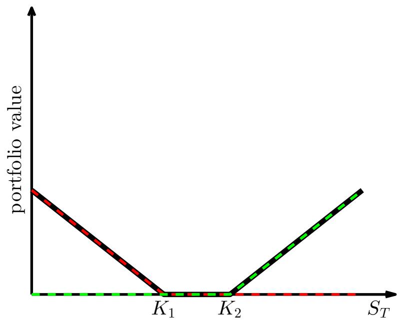
A speculator holding this portfolio would hope S_T would be away from K, and they would always do better if it moved above.

   (c) A put with strike K_1 and a call with strike $K_2 > K_1$ (a *strangle*).

The payoff is

$$(K_1 - S_T)^+ + (S_T - K_2)^+ = \begin{cases} K_1 - S_T & \text{if } S_T < K_1 \\ 0 & \text{if } K_1 \le S_T \le K_2 \\ S_T - K_2 & \text{if } S_T > K_2 \end{cases}$$





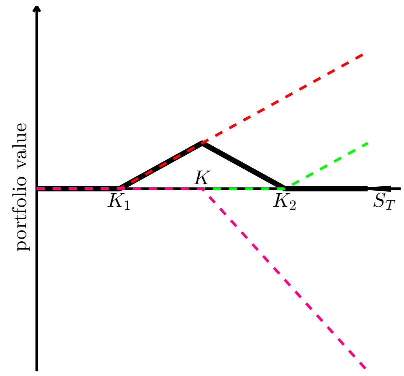
A speculator holding this portfolio hopes S_T will be more than $|K_2 - K_1|/2$ from $K = (K_1 + K_2)/2$.

(d) Long a call with strike K_1, long a call with strike $K_2 > K_1$ and short 2 calls with strike $K = (K_1 + K_2)/2$ (a *butterfly spread*).

The payoff is

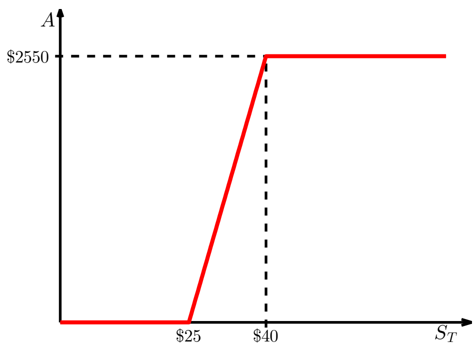
$$\left(\begin{array}{c} (S_T - K_1)^+ \\ + (S_T - K_2)^+ \\ - 2\,(S_T - K)^+ \end{array} \right) = \begin{cases} 0 & \text{if } S_T \le K_1 \\ S_T - K_1 & \text{if } K_1 < S_T \le K \\ K_2 - S_T & \text{if } K < S_T \le K_2 \\ 0 & \text{if } S_T > K_2 \end{cases}$$

A speculator holding this portfolio hopes S_T will be between K_1 and K_2 and would be delighted if was at K.



2. In 1986 Standard Oil issued bonds for which the bond holder received no interest but at maturity received the face value of the bond, \$1,000 (this is the amount lent by the bond holder to Standard Oil) plus an additional amount, A, which depended on the price of oil at the bond's maturity. This additional amount was set to be 170 times the excess, if any, of the price of a barrel of oil over \$25 subject to the condition that the additional amount, A, could not exceed \$2550.

   (a) Find a formula for A in terms of S_T, the price of a barrel of oil (in dollars per barrel) at the bond's maturity date T.
   $A = \min\big(\$2550, 170 \times (S_T - \$25)^+\big)$



   (b) Express the additional amount in terms of a long position in a number of call options with strike K_1 and a short position in a number of call options with strike K_2. (Specify the both the number and strikes of these call options, which are on the price of a barrel of oil.)
   A simple calculation shows that the cap of \$2550 kicks in when $S_T \geq \$40$. Therefore the two strikes are clearly $K_1 = \$25$ and $K_2 = \$40$. To get growth between \$25 and \$40 we need long calls

with strike $K_1 = \$25$ and to prevent further growth we need an equal number of short calls with strike $K_2 = \$40$. If we have only one of each, the maximum value is $A = \$15$, so we need $\$2550/\$15 = 170$ of each.

3. Let $c(S_t, t; K, T)$ denote the price of a call option with strike $K > 0$ and expiry T when the current time is $t < T$ and the share price is S_t. Use no arbitrage arguments to prove the following properties of European call prices.

   (a) $c(S_t, t; K, T) \leq S_t$.

   Suppose that $c(S_t, t; K, T) > S_t$. At t, write the call, buy the share and bank the profit. At t you have a bank balance of $c(S_t, t; K, T) - S_t > 0$. At expiry this will be worth

$$\big(c(S_t, t; K, T) - S_t\big)\, e^{r(T-t)} > 0.$$

   At expiry, if $0 < S_T \leq K$, the option will not be exercised so you just sell the share for S_T, increasing your profit to

$$S_T + \big(c(S_t, t; K, T) - S_t\big)\, e^{r(T-t)} > 0.$$

   If $S_T > K > 0$, the option is exercised so you receive the strike and deliver the share. This leaves you a profit of

$$K + \big(c(S_t, t; K, T) - S_t\big)\, e^{r(T-t)} > 0.$$

   In general at T your profit will be

$$\big(c(S_t, t; K, T) - S_t\big)\, e^{r(T-t)} + \min(S_T, K) > 0.$$

   There is zero risk to this strategy, it costs nothing to set up, it always results in a non-zero profit and so it is an arbitrage.

   (b) $c(S_t, t; K, T) \geq \max(S_t - K\, e^{-r(T-t)}, 0)$, where r is the risk-free rate.

   If the call has a non-positive price, $c(S_t, t; K, T) < 0$, go and buy it, you are *paid* a non-negative amount of money to do this. Call this $A = -c(S_t, t; K, T) > 0$. Put A in the bank until expiry when it will be worth $A\, e^{r(T-t)} > 0$.

   If $S_T \leq K$ the call will not be exercised, but you still have

$$A\, e^{r(T-t)} > 0$$

   in the bank. If $S_T > K$ the call is exercised, you pay K and get the share, so your profit is

$$A\, e^{r(T-t)} + \big(S_T - K\big) > 0.$$

   In general, your profit at T is

$$A\, e^{r(T-t)} + \max\big(S_T - K, 0\big).$$

This is a risk-free and strictly positive profit earned at no expense at time t and so represents an arbitrage. Thus $c(S_t, t; K, T) \geq 0$.

Suppose that $0 \leq c(S_t, t; K, T) < S_t - K\, e^{-r(T-t)}$, so rearranging

$$S_t - c(S_t, t; K, T) > K\, e^{-r(T-t)} > 0.$$

Short sell the share, buy the call and put the remaining cash,

$$A = S_t - c(S_t, t; K, T) > K\, e^{-r(T-t)},$$

in the bank. At expiry, if $0 < S_T \leq K$ the option is worthless but you have

$$A\, e^{r(T-t)} > K > S_T$$

in the bank, so buy back the share and close out short sale, leaving you with a profit of

$$A\, e^{r(T-t)} - S_T > 0.$$

If $S_T \geq K$ then you still have more than K in the bank, so exercise the option, pay K and receive the share then close out the short sale. Again, you still have

$$A\, e^{r(T-t)} - K > 0$$

in profit. In either event, you end up with a positive profit with zero risk and zero outlay. This is an arbitrage.

(c) if $0 < K_1 < K_2$ then

$$0 \leq c(S_t, t; K_1, T) - c(S_t, t; K_2, T) \leq (K_2 - K_1)\, e^{-r(T-t)}$$

where again r is the risk-free rate.

Denote the call with strike K_1 by C_1 and the call with strike K_2 by C_2. If $c(S_t, t; K_1, T) - c(S_t, t; K_2, T) < 0$, i.e., $C_1 - C_2 < 0$ or $C_1 < C_2$, then write C_2 and buy C_1. This generates a profit $A = C_2 - C_1 > 0$ at t which can be banked. At expiry this amount has grown to $A\, e^{r(T-t)} > 0$ and your position is

$$(S_T - K_1)^+ - (S_T - K_2)^+ = \begin{cases} 0 & \text{if } S_T \leq K_1, \\ S_T - K_1 & \text{if } K_1 < S_T \leq K_2, \\ K_2 - K_1 & \text{if } S_T > K_2. \end{cases}$$

Close inspection shows that this is non-negative no matter what $S_T > 0$ is. Thus you always end up with at least $A\, e^{r(T-t)} > 0$ and possibly more. This is an arbitrage and so we must have $C_1 \geq C_2$.

If $C_1 - C_2 > (K_2 - K_1)\, e^{-r(T-t)} > 0$, write C_1, buy C_2 and put the remaining funds, $A = C_1 - C_2 > (K_2 - K_1)\, e^{-r(T-t)} > 0$, in the bank. At expiry this will have grown to $K_2 - K_1 > 0$ and your net position is

$$(K_2 - K_1) + (S_T - K_2)^+ - (S_T - K_1)^+.$$

As above, the minimum possible value of $(S_T - K_2)^+ - (S_T - K_1)^+$ is $(K_1 - K_2) < 0$ and it occurs only if $S_T \geq K_2$. So if $S_T \geq K_2$ you end up with zero profit but if $0 < S_T < K_2$ you end up with a strictly positive profit. This is again an arbitrage.

(d) If $T_1 < T_2$ and $r > 0$, then $c(S_t, t; K, T_1) \leq c(S_t, t; K, T_2)$.
Denote the call with expiry T_1 by C_1 and the one with expiry T_2 by C_2. If $C_1 > C_2$, then write C_1 and buy C_2, which generates a positive amount of cash $A = C_1 - C_2 > 0$. At T_1 this is worth $A\, e^{r(T_1 - t)} > 0$. At T_1, C_1 is worth $(S_{T_1} - K)^+$ and C_2 is worth

$$c(S_{T_1}, T_1; K, T_2) \geq \left(S_{T_1} - K\, e^{-r(T_2 - T_1)}\right)^+ \geq \left(S_{T_1} - K\right)^+,$$

i.e., C_2 is worth no less than C_1. So, if necessary, sell C_2 to cover the cost of paying out the payoff for C_1 and your net profit is

$$A\, e^{r(T_1 - t)} + C_2 - C_1 \geq A\, e^{r(T_1 - t)} > 0.$$

This is another arbitrage.

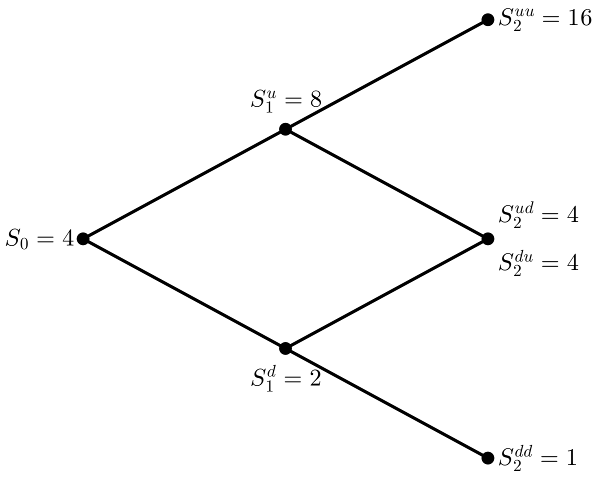


Figure 1: The share-price tree for Questions 4–6.

4. Consider a two-step binomial model in which at each step the share price either doubles, with probability $p \in (0,1)$, or halves, with probability $1-p \in (0,1)$. Initially the price is $S_0 = 4$. Assume each step takes one unit of time and that over one unit of time the risk-free rate is $r = \log(5/4)$. The possible prices are shown in Figure 1 below.

   (a) Show that the risk-neutral probability for an up-move is $q = \frac{1}{2}$.
   At any point inside the tree if the price is S_t then it can either move up to $S_{t+1}^u = 2\,S_t$ or down to $S_{t+1}^d = S_t/2$. Suppose that respective option prices are V_{t+1}^u and V_{t+1}^d. To determine the option price V_t and risk-neutral probabilities corresponding to S_t we can use a delta-hedging argument or a replication argument. I will use a replication argument. Hold ϕ_t shares and ψ_t in cash, so the replicating portfolio's value at t is

$$\Phi_t = \phi_t\,S_t + \psi_t.$$

   At time $t+1$ it is one of

$$\begin{aligned} \Phi_{t+1}^u &= \phi_t\,S_{t+1}^u + \tfrac{5}{4}\psi_t &= 2\,S_t\,\phi_t + \tfrac{5}{4}\psi_t, \\ \Phi_{t+1}^d &= \phi_t\,S_{t+1}^d + \tfrac{5}{4}\psi_t &= \tfrac{1}{2}\,S_t\,\phi_t + \tfrac{5}{4}\psi_t. \end{aligned}$$

   Set these equal to V_{t+1}^u and V_{t+1}^d, respectively so

$$\begin{aligned} 2\,S_t\,\phi_t + \tfrac{5}{4}\,\psi_t &= V_{t+1}^u, \\ \tfrac{1}{2}\,S_t\,\phi_t + \tfrac{5}{4}\,\psi_t &= V_{t+1}^d, \end{aligned} \qquad \begin{pmatrix} 2\,S_t & \frac{5}{4} \\ \frac{1}{2}\,S_t & \frac{5}{4} \end{pmatrix} \begin{pmatrix} \phi_t \\ \psi_t \end{pmatrix} = \begin{pmatrix} V_{t+1}^u \\ V_{t+1}^d \end{pmatrix}$$

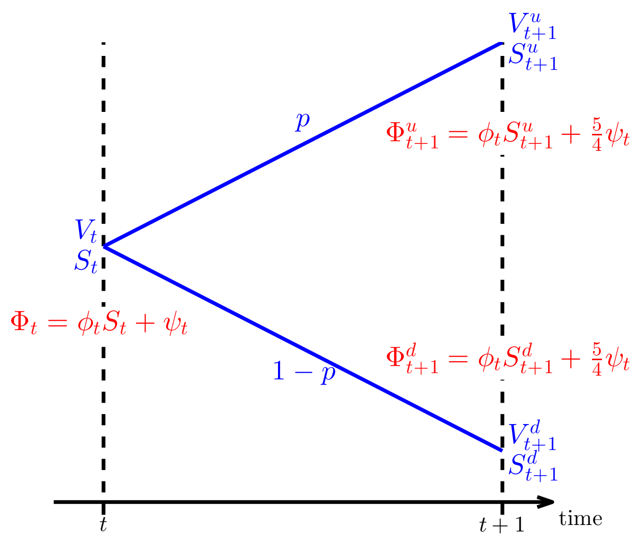
solve for ϕ_t and ψ_t

$$\begin{aligned}
\begin{pmatrix} \phi_t \\ \psi_t \end{pmatrix} &= \begin{pmatrix} 2\,S_t & \frac{5}{4} \\ \frac{1}{2}\,S_t & \frac{5}{4} \end{pmatrix}^{-1} \begin{pmatrix} V_{t+1}^u \\ V_{t+1}^d \end{pmatrix} \\
&= \frac{8}{15\,S_t} \begin{pmatrix} \frac{5}{4} & -\frac{5}{4} \\ -\frac{1}{2}\,S_t & 2\,S_t \end{pmatrix} \begin{pmatrix} V_{t+1}^u \\ V_{t+1}^d \end{pmatrix} \\
&= \begin{pmatrix} \frac{2}{3}\big(V_{t+1}^u - V_{t+1}^d\big)/S_t \\ \frac{4}{15}\big(4\,V_{t+1}^d - V_{t+1}^u\big) \end{pmatrix}
\end{aligned}$$

and then show that

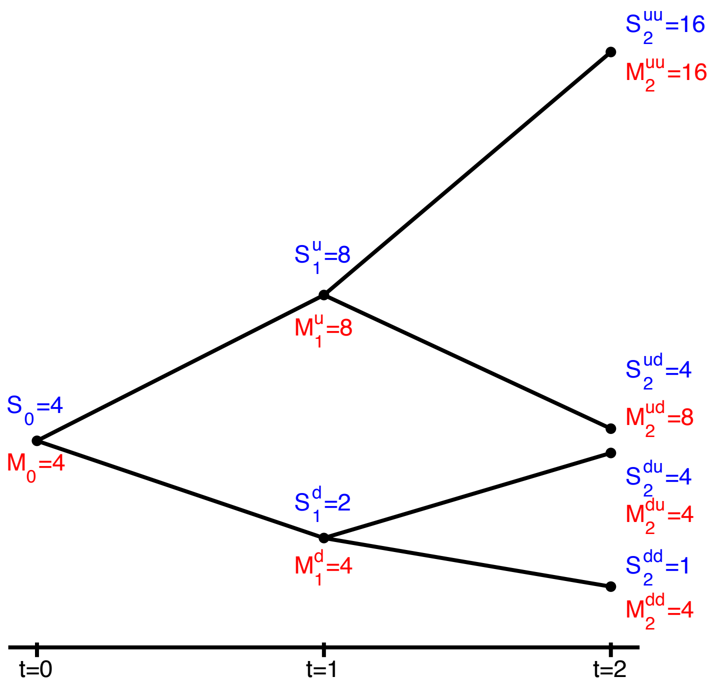
$$\begin{aligned}
\Phi_t &= \phi_t\,S_t + \psi_t \\
&= \tfrac{2}{5}\big(V_{t+1}^u + V_{t+1}^d\big) \\
&= \tfrac{4}{5}\big(\tfrac{1}{2}V_{t+1}^u + \tfrac{1}{2}V_{t+1}^d\big) \\
&= e^{-r}\big(q\,V_{t+1}^u + (1-q)\,V_{t+1}^d\big),
\end{aligned}$$

with $q = \frac{1}{2}$ and $e^r = \frac{5}{4}$.
You can also find q using a Δ-hedging argument.



(b) Suppose now, and for the rest of the question, that the option has a payoff which depends on the maximum share price over the life of the option, given by

$$Y = \max_{t=0,1,2} \big(S_t - 6\big)^+.$$



(This means it is a fixed-strike lookback call.) Compute the final option values $V_2^\omega = Y^\omega$ for each of the outcomes, i.e., possible paths, $\omega \in \Omega = \{uu, ud, du, dd\}$.

Along the paths we find that the maximum value of S and hence Y are

$$\begin{array}{lll} \{uu\} & M_2^{uu} = 16 & Y^{uu} = 10, \\ \{ud\} & M_2^{ud} = 8 & Y^{ud} = 2, \\ \{du\} & M_2^{du} = 4 & Y^{du} = 0, \\ \{dd\} & M_2^{dd} = 1 & Y^{dd} = 0. \end{array}$$

Thus

$$\begin{aligned} V_2^{uu} &= Y^{uu} &= 10, \\ V_2^{ud} &= Y^{ud} &= 2, \\ V_2^{du} &= Y^{du} &= 0, \\ V_2^{dd} &= Y^{dd} &= 0. \end{aligned}$$

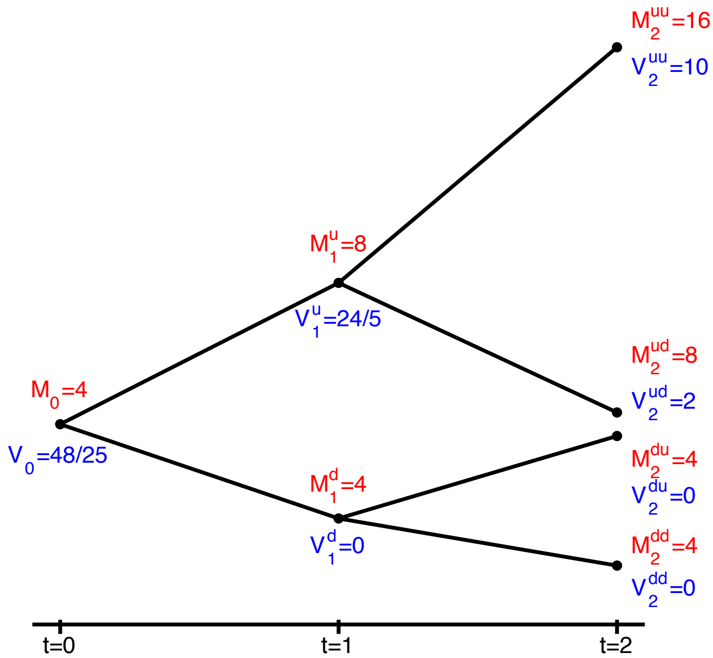
(c) Compute the values of V_1^u and V_1^d, and hence find V_0.

The risk-neutral pricing formula gives

$$\begin{aligned} V_1^u &= \tfrac{2}{5}\left(V_2^{uu} + V_2^{ud}\right) &= \tfrac{24}{5} = 4.8, \\ V_1^d &= \tfrac{2}{5}\left(V_2^{du} + V_2^{dd}\right) &= 0, \end{aligned}$$

from which we can find

$$V_0 = \tfrac{2}{5}\left(V_1^u + V_1^d\right) = \tfrac{48}{25} = 1.92.$$



(d) Find the replicating portfolios $(\phi_1^\omega, \psi_1^\omega)$ for $\omega \in \{u, d\}$.

One way to do this is to compute

$$\phi_1^\omega = \frac{V_2^{\omega\, u} - V_2^{\omega\, d}}{S_2^{\omega\, u} - S_2^{\omega\, d}}, \quad \psi_1^\omega = V_1^\omega - \phi_1^\omega\, S_1^\omega,$$

rather than use the complicated formula for ψ_1^ω directly. This gives

$$\begin{aligned} \phi_1^u &= \tfrac{10-2}{16-4} &= \tfrac{2}{3}, \\ \phi_1^d &= \tfrac{0-0}{4-1} &= 0, \end{aligned}$$

and

$$\begin{aligned} \psi_1^u &= \tfrac{24}{5} - \tfrac{2}{3}\, 8 &= -\tfrac{8}{15}, \\ \psi_1^d &= \quad 0. \end{aligned}$$

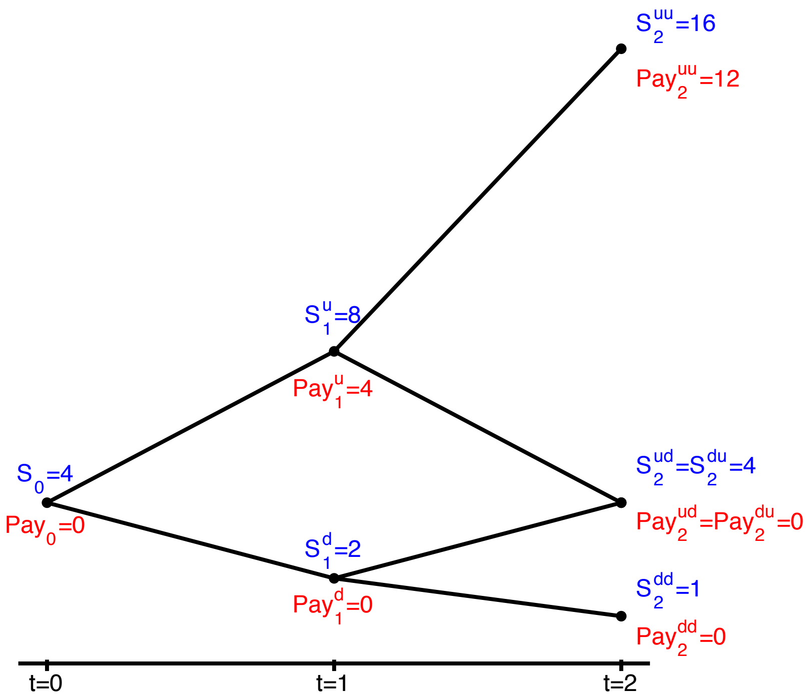
(e) Find the replicating portfolio (ϕ_0, ψ_0).

As above, we find that

$$\phi_0 = \tfrac{24/5-0}{8-2} = \tfrac{4}{5}$$

and

$$\psi_0 = V_0 - \phi_0\, S_0 = \tfrac{48}{25} - 4 \times \tfrac{4}{5} = -\tfrac{32}{25}.$$



5. Consider a two-step binomial model for a share price with the same properties as in Question 4. An American call is written on this share, with strike $K = 4$. Compute the prices of this American call on the tree and show that they equal the corresponding prices of a European call with strike $K = 4$.

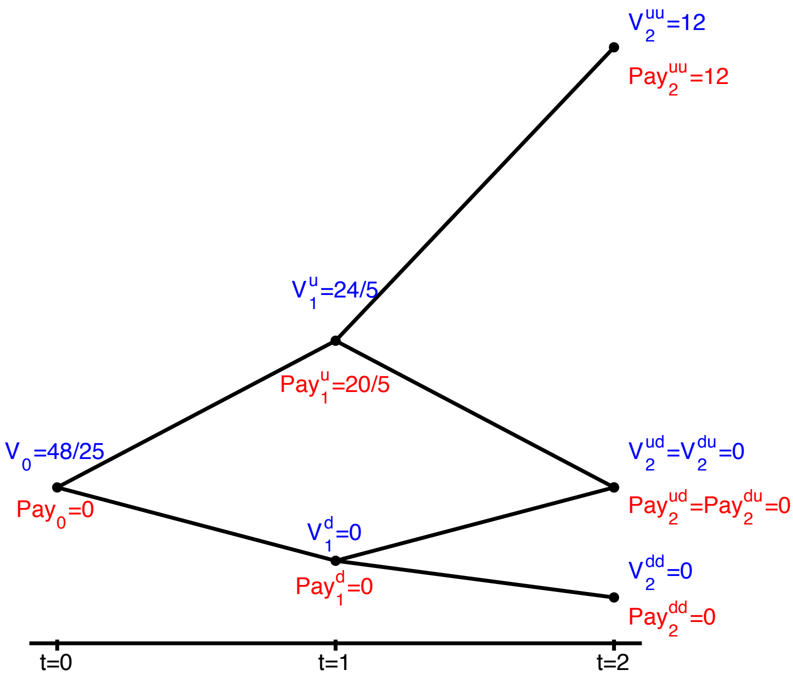
   The difference between an American option and a European option is that the American option can be exercised at any time $t = 0$, $t = 1$ or $t = 2$, while the European can only be exercised at time $t = 2$. The only reason we would have to exercise the American option at times $t = 1$ or $t = 0$ is if the payoff is more valuable than the option itself. If we compute the European value and it is always more valuable than the payoff (at the same point) then it is never optimal to exercise the option prior to expiry and the American and European options must have the same values.

   In this case we have to compute the payoff, $\max(S-K, 0)$, at all points on the tree. Note that this only depends on the share price at each point, not how we got to that point, so we can use a recombining tree to do so.

   At expiry we have the option price given by the payoff, so

$$V_2^{uu} = 12, \quad V_2^{ud} = V_2^{du} = 0, \quad V_2^{dd} = 0.$$

   This is the same whether the option is American or European.



For $t = 1$, using the formula

$$V_t^\omega = e^{-r}\left(q\, V_{t+1}^{\omega\, u} + (1-q)\, V_{t+1}^{\omega\, d}\right) = \tfrac{2}{5}\left(V_{t+1}^{\omega\, u} + V_{t+1}^{\omega\, d}\right)$$

for the price of a European option we find that

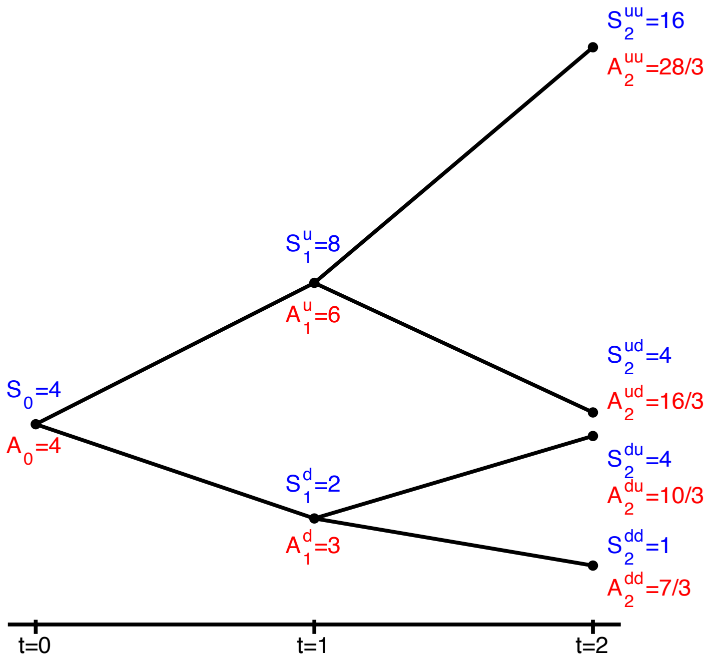
$$\begin{aligned} V_1^u &= \tfrac{2}{5}\big(12+0\big) = 4\,\tfrac{4}{5} > \mathrm{Pay}_1^u = 4, \\ V_1^d &= \tfrac{2}{5}\big(0+0\big) = 0 \geq \mathrm{Pay}_1^d = 0, \end{aligned}$$

which shows that there is no point exercising at time $t = 1$; the pay-off value is never greater than the European option value. Thus the American and European option values are the same at $t = 1$.

By the same reasoning

$$V_0 = \tfrac{2}{5}\big(V_1^u + V_1^d\big) = \tfrac{2}{5}\big(\tfrac{24}{5}+0\big) = \tfrac{48}{25} = 1.92 > \mathrm{Pay}_0 = 0$$

so there is no point exercising the American option at time $t = 0$. Again, the European and American call option have the same value at $t = 0$.



6. With the same two-step binomial model as in the previous two questions, consider the following American option written on the stock. If the option is exercised at time $t \in 0, 1, 2$ it pays out

$$Y_t = \left(\frac{1}{1+t} \Big(\sum_{k=0}^{t} S_k \Big) - 2 \right)^+,$$

i.e., it is an American call with fixed strike $K = 2$ on the *average* share price at time t. Find the value of the option on all paths through the tree and determine the optimal exercise strategy (i.e., find the points on paths in the tree where it is optimal to exercise the option).

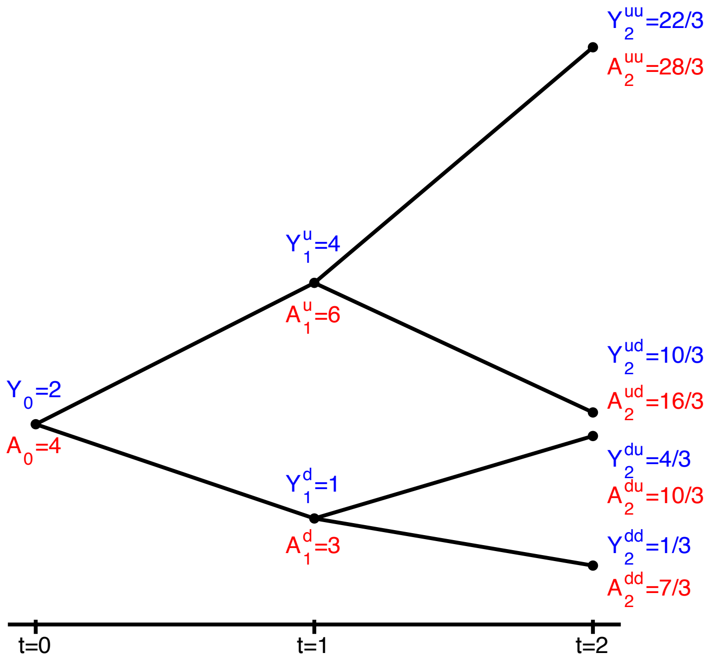
The first thing to do is to compute the average prices, $A_t = \frac{1}{1+t}\Big(\sum_{k=0}^{t} S_k\Big)$, at all points on the tree. The averages do depend on the price paths and so it is best to not use a recombining tree.

The values of the averages are

$$A_2^{dd} = \tfrac{7}{3},\ A_2^{du} = \tfrac{10}{3},\ A_2^{ud} = \tfrac{16}{3},\ A_2^{uu} = \tfrac{28}{3},$$
$$A_1^d = 3, \quad A_1^u = 6,$$
$$A_0 = 4.$$

Once this is done, we can compute the payoff

$$Y_t = \left(\frac{1}{1+t} \Big(\sum_{k=0}^{t} S_k \Big) - 2 \right)^+$$



at all points on the tree and set $V_2^\omega = Y_2^\omega$ at expiry. The values of Y are

$$Y_2^{dd} = \tfrac{1}{3},\ Y_2^{du} = \tfrac{4}{3},\ Y_2^{ud} = \tfrac{10}{3},\ Y_2^{uu} = \tfrac{22}{3},$$
$$Y_1^d = 1, \quad Y_1^u = 4,$$
$$Y_0 = 2$$

and therefore

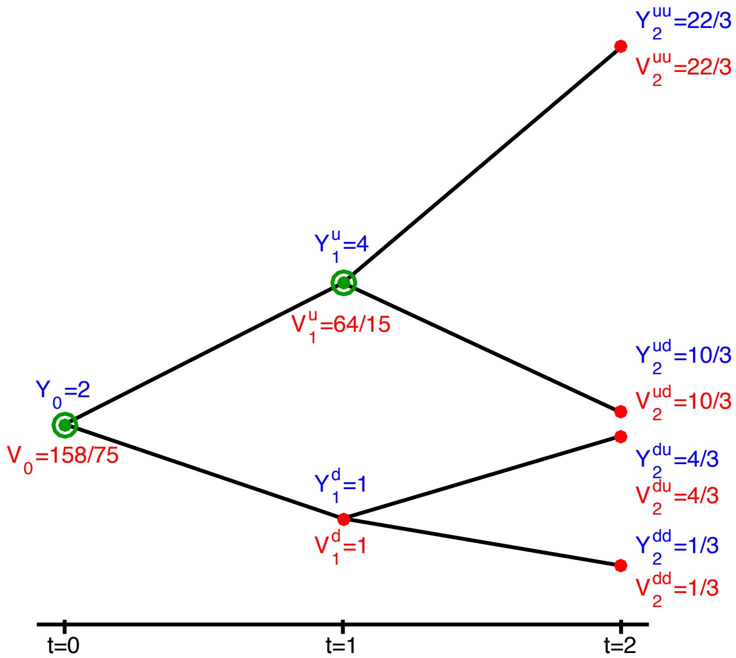
$$V_2^{dd} = \tfrac{1}{3}, \quad V_2^{du} = \tfrac{4}{3}, \quad V_2^{ud} = \tfrac{10}{3}, \quad V_2^{uu} = \tfrac{22}{3}.$$

We can compute the European option values, say U_1^u and U_1^d, as

$$\begin{aligned} U_1^d &= \tfrac{2}{5}\left(V_2^{dd} + V_2^{du}\right) = \tfrac{2}{5}\left(\tfrac{1}{3} + \tfrac{4}{3}\right) = \tfrac{2}{3} < Y_1^d = 1, \\ U_1^u &= \tfrac{2}{5}\left(V_2^{ud} + V_2^{uu}\right) = \tfrac{2}{5}\left(\tfrac{10}{3} + \tfrac{22}{3}\right) = \tfrac{64}{15} > Y_1^u = 4. \end{aligned}$$

In the first case the payoff Y_1^d is more valuable than holding the option U_1^d and so it makes sense to exercise at this point. In the second case the payoff Y_1^u is less valuable than holding the option U_1^u and so it makes sense to hold the option at this point. Therefore the American option values at $t = 1$ are

$$\begin{aligned} V_1^d &= \max\left(Y_1^d, U_1^d\right) = Y_1^d = 1, \\ V_1^u &= \max\left(Y_1^u, U_1^u\right) = U_1^u = \tfrac{64}{15}. \end{aligned}$$



Similarly, we compute the European option price at $t = 0$ as

$$U_0 \;=\; \tfrac{2}{5}\left(V_1^d + V_1^u\right) \;=\; \tfrac{2}{5}\left(1 + \tfrac{65}{15}\right) \;=\; \tfrac{158}{75} \;>\; Y_0 = 2.$$

Note that we use V_1^u and V_1^d here, not U_1^u and U_1^d. As the option value U_0 is more valuable than the payoff Y_0, we don't exercise, i.e.,

$$V_0 \;=\; \max\big(Y_0, U_0\big) \;=\; U_0 \;=\; \tfrac{158}{75}.$$

Thus the only points at which we don't exercise are at $t = 0$ and at $t = 1$ in the up-state, as shown in the final diagram; a (green) circle around a point indicates that the option should *not* be exercised at this point.