

B8.3 Mathematical Models for Financial Derivatives

Hilary Term 2019

Problem Sheet Two

In the following $(W_t)_{t \geq 0}$ denotes a standard Brownian motion and $t \geq 0$ denotes time. A partition π of the interval $[0, t]$ is a sequence of points $0 = t_0 < t_1 < t_2 < \dots < t_n = t$ and $|\pi| = \max_k(t_{k+1} - t_k)$. On a given partition $W_k \equiv W_{t_k}$, $\delta W_k \equiv W_{k+1} - W_k$, $\delta t_k \equiv t_{k+1} - t_k$ and if f is a function on $[0, t]$, $f_k \equiv f(t_k)$ and $\delta f_k \equiv f_{k+1} - f_k$.

1. Show that if $t, s \geq 0$ then $\mathbb{E}[W_s W_t] = \min(s, t)$.
2. Suppose we define the following two stochastic integrals, the ‘anti-Itô’ integral

$$\int_0^t f(W_s, s) \bullet dW_s = \lim_{|\pi| \rightarrow 0} \sum_{k=0}^{n-1} f(W_{k+1}, t_{k+1})(W_{k+1} - W_k),$$

and the Stratonovich integral

$$\begin{aligned} \int_0^t f(W_s, s) \circ dW_s \\ = \lim_{|\pi| \rightarrow 0} \sum_{k=0}^{n-1} \frac{1}{2} (f(W_{k+1}, t_{k+1}) + f(W_k, t_k)) (W_{k+1} - W_k). \end{aligned}$$

Show that

$$\begin{aligned} 2 \int_0^t W_s \bullet dW_s &= W_t^2 + t, & \mathbb{E} \left[2 \int_0^t W_s \bullet dW_s \right] &= 2t, \\ 2 \int_0^t W_s \circ dW_s &= W_t^2, & \mathbb{E} \left[2 \int_0^t W_s \circ dW_s \right] &= t \end{aligned}$$

3. Assuming that both the integral and its variance exist, show that

$$\text{var} \left[\int_0^t f(W_s, s) dW_s \right] = \int_0^t \mathbb{E}[f(W_s, s)^2] ds.$$

[Note: if the integral and its variance exist then it is legitimate to interchange the order of expectation and integration.]

4. Use the differential version of Itô’s lemma to show that

$$(a) \int_0^t W_s ds = t W_t - \int_0^t s dW_s = \int_0^t (t - s) dW_s,$$

$$(b) \int_0^t W_s^2 dW_s = \frac{1}{3} W_t^3 - \int_0^t W_s ds,$$

$$(c) \mathbb{E}[e^{aW_t - a^2 t/2}] = 1.$$

5. Define X_t to be the ‘area under a Brownian motion’, $X_0 = 0$ and $X_t = \int_0^t W_u du$ for $t > 0$. Show that X_t is normally distributed with

$$\mathbb{E}[X_t] = 0, \quad \mathbb{E}[X_t^2] = \frac{1}{3} t^3.$$

Now define Y_t as the ‘average area under a Brownian motion’,

$$Y_t = \begin{cases} 0 & \text{if } t = 0, \\ X_t/t & \text{if } t > 0. \end{cases}$$

Show that Y_t has $\mathbb{E}[Y_t] = 0$, $\mathbb{E}[Y_t^2] = t/3$ and that Y_t is continuous for all $t \geq 0$.

Is $\sqrt{3} Y_t$ a Brownian motion? Give reasons for your answer.

6. Show that if

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t,$$

and $S_t^* = f(S_t) = S_t^3$, then

$$\frac{dS_t^*}{S_t^*} = 3(\mu + \sigma^2) dt + 3\sigma dW_t.$$

7. Find solutions of the Black-Scholes terminal value problem

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - y) S \frac{\partial V}{\partial S} - r V = 0, \quad S > 0, \quad t < T,$$

$$V(S, T) = f(S), \quad S > 0,$$

when

- (a) $f(S) = C$, where C is a constant;
- (b) $f(S) = S^\alpha$, where α is a constant.

[Hint: you don’t need the Feynman-K ac formula to do either of these. Look for simple functional forms of the solution.]

Optional questions

8. The Black-Scholes equation from a binomial method.

One step of the Cox, Ross & Rubinstein binomial method can be written as

$$V(S, t) = e^{-r\delta t} (q V^u + (1 - q) V^d)$$

where

$$V^u = V(S^u, t + \delta t), \quad V^d = V(S^d, t + \delta t),$$

$$S^u = S e^{\sigma\sqrt{\delta t}}, \quad S^d = S e^{-\sigma\sqrt{\delta t}}, \quad q = \frac{e^{r\delta t} - e^{-\sigma\sqrt{\delta t}}}{e^{\sigma\sqrt{\delta t}} - e^{-\sigma\sqrt{\delta t}}},$$

$\sigma > 0$ is the volatility, r is the risk-free rate and $\delta t > 0$ is the length of the time-step. Supposing this is true for all $S > 0$ and that $V(S, t)$ may be expanded in a Taylor series in both S and t , show that as $\delta t \rightarrow 0$

$$\begin{aligned} q &= \frac{1}{2} + \frac{r - \frac{1}{2}\sigma^2}{2\sigma} \sqrt{\delta t} + \mathcal{O}(\delta t), \\ V^u &= V + \sqrt{\delta t} \sigma S \frac{\partial V}{\partial S} + \delta t \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 \left(S \frac{\partial V}{\partial S} + S^2 \frac{\partial^2 V}{\partial S^2} \right) \right) + \mathcal{O}(\delta t^{3/2}), \\ V^d &= V - \sqrt{\delta t} \sigma S \frac{\partial V}{\partial S} + \delta t \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 \left(S \frac{\partial V}{\partial S} + S^2 \frac{\partial^2 V}{\partial S^2} \right) \right) + \mathcal{O}(\delta t^{3/2}), \end{aligned}$$

where V and all its partial derivatives are evaluated at (S, t) . Hence show that in the limit $\delta t \rightarrow 0$ the option price satisfies the (zero dividend-yield) Black-Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r S \frac{\partial V}{\partial S} - r V = 0.$$

9. The variation of a function, or stochastic process, over $[0, t]$, is

$$\langle f \rangle_t = \lim_{|\pi| \rightarrow 0} \sum_{k=0}^{n-1} |f_{k+1} - f_k|.$$

If $\langle f \rangle_t$ is finite on $[0, t]$ we say f has bounded variation on $[0, t]$. Show that:

- (a) if f is $C^1[0, t]$ then $\langle f \rangle_t = \int_0^t |f'(t)| dt < \infty$;
- (b) if f is a continuous function with $\langle f \rangle_t < \infty$ then its quadratic variation is zero, $[f]_t = 0$;
- (c) Brownian motion does not have bounded variation;
- (d) the arc length of the graph of a Brownian motion is infinite for any $t > 0$.

[Hint: if $y = X_t$ has an arc length s then $ds = \sqrt{dy^2 + dx^2} \geq \sqrt{dy^2} = |dy|$.]