B8.3 Mathematical Models for Financial Derivatives

Hilary Term 2019

Solution Sheet Two

In the following $(W_t)_{t\geq 0}$ denotes a standard Brownian motion and $t \geq 0$ denotes time. A partition π of the interval $[0,t]$ is a sequence of points $0 = t_0 < t_1 < t_2 < \cdots < t_n = t$ and $|\pi| = \max_k(t_{k+1} - t_k)$. On a given partition $W_k \equiv W_{t_k}$, $\delta W_k \equiv W_{k+1} - W_k$, $\delta t_k \equiv t_{k+1} - t_k$ and if f is a function on $[0,t]$, $f_k \equiv f(t_k)$ and $\delta f_k \equiv f_{k+1} - f_k$.

1. Show that if $t, s \geq 0$ then $\mathbb{E}[W_s\, W_t] = \min(s,t)$.

   If $s = t \geq 0$ then we have

$$\mathbb{E}[W_s\, W_t] = \mathbb{E}[W_t^2] = t.$$

   If not, assume $0 \leq s < t$ and write $W_t = W_s + W_t - W_s$. Then

$$\begin{aligned}\mathbb{E}[W_s\, W_t] &= \mathbb{E}\big[W_s\, W_s + W_s\,(W_t - W_s)\big]\\ &= \mathbb{E}[W_s\, W_s] + \mathbb{E}\big[W_s\,(W_t - W_s)\big]\\ &= \mathbb{E}[W_s^2] \;=\; s \;=\; \min(s,t).\end{aligned}$$

2. Suppose we define the following two stochastic integrals, the ‘anti-Itô’ integral

$$\int_0^t f(W_s, s) \bullet dW_s \;=\; \lim_{|\pi|\to 0} \sum_{k=0}^{n-1} f(W_{k+1}, t_{k+1})(W_{k+1} - W_k),$$

   and the Stratonovich integral

$$\int_0^t f(W_s, s) \circ dW_s$$
$$= \lim_{|\pi|\to 0} \sum_{k=0}^{n-1} \tfrac{1}{2}\big(f(W_{k+1}, t_{k+1}) + f(W_k, t_k)\big)\,(W_{k+1} - W_k).$$

   Show that

$$\begin{aligned}2\int_0^t W_s \bullet dW_s &= W_t^2 + t, &\quad \mathbb{E}\Big[2\int_0^t W_s \bullet dW_s\Big] &= 2\,t,\\ 2\int_0^t W_s \circ dW_s &= W_t^2, &\quad \mathbb{E}\Big[2\int_0^t W_s \circ dW_s\Big] &= t\end{aligned}$$

$$\begin{aligned}
\int_0^t 2W_s \bullet dW_s &= \lim_{|\pi|\to 0} \sum_{k=0}^{n-1} 2W_{k+1}\left(W_{k+1}-W_k\right) \\
&= \lim_{|\pi|\to 0} \sum_{k=0}^{n-1} \left(W_{k+1}^2 - W_k^2\right) + \lim_{|\pi|\to 0} \sum_{k=0}^{n-1} \left(W_{k+1}-W_k\right)^2 \\
&= W_n^2 - W_0^2 + \lim_{|\pi|\to 0} \sum_{k=0}^{n-1} \left(W_{k+1}-W_k\right)^2 \\
&= W_t^2 + t.
\end{aligned}$$

Since we know $\mathbb{E}\big[W_t^2\big] = t$, we see that

$$\mathbb{E}\Big[\int_0^t 2W_s \bullet dW_s\Big] = \mathbb{E}\big[W_t^2\big] + t = 2\,t.$$

Similarly we have

$$\begin{aligned}
\int_0^t 2W_s \circ dW_s &= \lim_{|\pi|\to 0} \sum_{k=0}^{n-1} \left(W_{k+1}+W_k\right)\left(W_{k+1}-W_k\right) \\
&= \lim_{|\pi|\to 0} \sum_{k=0}^{n-1} \left(W_{k+1}^2 - W_k^2\right) \\
&= W_n^2 - W_0^2 \\
&= W_t^2,
\end{aligned}$$

and so

$$\mathbb{E}\Big[\int_0^t 2W s \circ dW_s\Big] = \mathbb{E}\big[W_t^2\big] = t.$$

3. Assuming that both the integral and its variance exist, show that

$$\operatorname{var}\Big[\int_0^t f(W_s, s)\, dW_s\Big] = \int_0^t \mathbb{E}\big[f(W_s, s)^2\big]\, ds.$$

[Note: if the integral and its variance exist then it is legitimate to interchange the order of expectation and integration.]

If the integral exists then its expected value is zero so

$$\operatorname{var}\Big[\int_0^t f(W_s, s)\, dW_s\Big] = \mathbb{E}\Big[\Big(\int_0^t f(W_s, s)\, dW_s\Big)^2\Big].$$

Set $f_k = f(W_k, t_k)$ and consider

$$\begin{aligned}
\Big(\sum_{k=0}^{n-1} f_k\, \delta W_k\Big)^2 &= \sum_{k=0}^{n-1}\sum_{j=0}^{n-1} f_j\, f_k\, \delta W_j\, \delta W_k \\
&= \sum_{k=0}^{n-1} f_k^2\, (\delta W_k)^2 + 2\sum_{k=1}^{n-1}\sum_{j<k} f_j\, f_k\, \delta W_j\, \delta W_k.
\end{aligned}$$

We can eliminate the double sum using the tower law,

$$\begin{aligned}\mathbb{E}\Big[f_j\, f_k\, \delta W_j\, \delta W_k\Big] &= \mathbb{E}\Big[\mathbb{E}_{t_k}\big[f_j\, f_k\, \delta W_j\, \delta W_k\big]\Big] \\ &= \mathbb{E}\Big[f_j\, f_k\, \delta W_j\, \mathbb{E}_{t_k}\big[\delta W_k\big]\Big]\end{aligned}$$

since $t_j < t_k$, which means f_j and δW_j are known at time t_k, as is $f_k = f(W_k, t_k)$. As $\mathbb{E}_{t_k}\big[\delta W_k\big] = 0$, this term vanishes, leaving

$$\begin{aligned}\mathbb{E}\Big[\Big(\sum_{k=0}^{n-1} f_k\, \delta W_k\Big)^2\Big] &= \mathbb{E}\Big[\sum_{k=0}^{n-1} f_k^2\, (\delta W_k)^2\Big] \\ &= \mathbb{E}\Big[\sum_{k=0}^{n-1} \mathbb{E}_{t_k}\big[f_k^2\, (\delta W_k)^2\big]\Big] \\ &= \mathbb{E}\Big[\sum_{k=0}^{n-1} f_k^2\, \mathbb{E}_{t_k}\big[(\delta W_k)^2\big]\Big] \\ &= \mathbb{E}\Big[\sum_{k=0}^{n-1} f_k^2\, \delta t_k\Big] = \sum_{k=0}^{n-1} \mathbb{E}\big[f_k^2\big]\, \delta t_k\end{aligned}$$

and as we refine the partition, $|\pi| \to 0$, we see

$$\lim_{|\pi|\to 0} \sum_{k=0}^{n-1} \mathbb{E}\big[f_k^2\big]\, \delta t_k \to \int_0^t \mathbb{E}\Big[f(W_s, s)^2\Big]\, ds.$$

4. Use the differential version of Itô's lemma to show that

(a) $\displaystyle\int_0^t W_s\, ds = t\, W_t - \int_0^t s\, dW_s = \int_0^t (t-s)\, dW_s,$

We have

$$d\big(t\, W_t\big) = W_t\, dt + t\, dW_t$$

which integrates to show

$$t\, W_t = \int_0^t W_s\, ds + \int_0^t s\, dW_s.$$

Rearranging gives

$$\int_0^t W_s\, ds = t\, W_t - \int_0^t s\, dW_s = \int_0^t (t-s)\, dW_s.$$

(b) $\displaystyle\int_0^t W_s^2\,dW_s = \tfrac{1}{3}W_t^3 - \int_0^t W_s\,ds,$

This time

$$d\big(W_t^3\big) = 3\,W_t^2\,dW_t + 3\,W_t\,dt$$

which integrates to give

$$W_t^3 = 3\int_0^t W_s^2\,dW_s + 3\int_0^t W_s\,ds.$$

Dividing by 3 and rearranging gives

$$\int_0^t W_s^2\,dW_s = \tfrac{1}{3}\,W_t^3 - \int_0^t W_s\,ds.$$

(c) $\mathbb{E}\big[e^{aW_t - a^2t/2}\big] = 1.$

Set $f(W,t) = \exp(aW - \tfrac{1}{2}a^2t)$ and $f_t = f(W_t,t)$. Itô's lemma gives

$$\begin{aligned} df_t &= -\tfrac{1}{2}\,a^2\,f_t\,dt + a\,f_t\,dW_t + \tfrac{1}{2}\,a^2\,f_t\,dt \\ &= a\,f_t\,dW_t. \end{aligned}$$

This integrates to give $f_t - f_0 = a\displaystyle\int_0^t f_s\,dW_s$ or, in full,

$$\exp\big(aW_t - \tfrac{1}{2}a^2t\big) = 1 + a\int_0^t \exp\big(aW_s - \tfrac{1}{2}a^2s\big)\,dW_s$$

Taking expectations gives

$$\begin{aligned} \mathbb{E}\Big[\exp\big(aW_t - \tfrac{1}{2}a^2t\big)\Big] &= 1 + a\,\mathbb{E}\Big[\int_0^t \exp\big(aW_s - \tfrac{1}{2}a^2s\big)\,dW_s\Big] \\ &= 1. \end{aligned}$$

5. Define X_t to be the 'area under a Brownian motion', $X_0 = 0$ and $X_t = \int_0^t W_u\,du$ for $t > 0$. Show that X_t is normally distributed with

$$\mathbb{E}\big[X_t\big] = 0, \quad \mathbb{E}\big[X_t^2\big] = \tfrac{1}{3}\,t^3.$$

From Question 4(a) we have

$$\int_0^t W_s\,ds = \int_0^t (t-s)\,dW_s.$$

As $t - s$ does not depend on W_s, the integral is normally distributed with

$$\mathbb{E}\Big[\int_0^t W_s\,ds\Big] = \mathbb{E}\Big[\int_0^t (t-s)\,dW_s\Big] = 0$$

and

$$\begin{aligned}\mathrm{var}\Big[\int_0^t W_s\,ds\Big] &= \mathrm{var}\Big[\int_0^t (t-s)\,dW_s\Big] \\ &= \int_0^t (t-s)^2\,ds \\ &= \tfrac{1}{3}\,t^3.\end{aligned}$$

Note that X_t is continuously differentiable for $t > 0$, with $\dot{X}_t = W_t$ (recall W_t is continuous in t).

Now define Y_t as the 'average area under a Brownian motion',

$$Y_t = \begin{cases} 0 & \text{if } t = 0, \\ X_t/t & \text{if } t > 0. \end{cases}$$

Show that Y_t has $\mathbb{E}[Y_t] = 0$, $\mathbb{E}\big[Y_t^2\big] = t/3$ and that Y_t is continuous for all $t \geq 0$.

Is $\sqrt{3}\,Y_t$ a Brownian motion? Give reasons for you answer.

For any $t > 0$, Y_t is normal because X_t is, with

$$\mathrm{E}\big[Y_t\big] = \mathrm{E}\big[X_t\big]/t = 0$$

and

$$\mathrm{var}\big[Y_t\big] = \mathrm{var}\big[X_t/t\big] = \mathrm{var}\big[X_t\big]/t^2 = \tfrac{1}{3}\,t.$$

For $t > 0$ we have $Y_t = X_t/t$ which is the ratio of two differentiable functions and as the denominator is never zero it follows that Y_t is differentiable for $t > 0$, which implies continuous. Moreover, it means we can use l'Hopital's rule to show

$$\lim_{t \to 0^+} Y_t = \lim_{t \to 0^+} \frac{W_t}{1} = 0,$$

which shows that Y_t is continuous at $t = 0$.

The function $\sqrt{3}\,Y_t$ is not only continuous but *differentiable* for $t > 0$ and therefore it can't be a Brownian motion. (The hard way to do this part of the question is to show that the increments over disjoint intervals are not independent.)

6. Show that if

$$\frac{dS_t}{S_t} = \mu\,dt + \sigma\,dW_t,$$

and $f_t = f(S_t) = S_t^3$, then

$$\frac{df_t}{f_t} = 3\,(\mu + \sigma^2)\,dt + 3\,\sigma\,dW_t.$$

Write the SDE for S_t as

$$dS_t = \mu\,S_t\,dt + \sigma\,S_t\,dW_t$$

so Itô's lemma for a general $F_t = F(S_t, t)$ is

$$dF_t = \left(\frac{\partial F}{\partial t}(S_t, t) + \tfrac{1}{2}\sigma^2\,S_t^2\,\frac{\partial^2 F}{\partial S^2}(S_t, t)\right)dt + \frac{\partial F}{\partial S}(S_t, t)\,dS_t.$$

With $f(S) = S^3$, so $f'(S) = 3\,S^2$ and $f''(S) = 6\,S$, this becomes

$$\begin{aligned} df_t &= 3\,\sigma^2\,S_t^3\,dt + 3\,S_t^2\,dS_t \\ &= 3\left(\sigma^2\,f_t\,dt + S_t^2\,(\mu\,S_t\,dt + \sigma\,S_t\,dW_t)\right) \\ &= 3\,f_t\,((\mu + \sigma^2)\,dt + \sigma\,dW_t) \end{aligned}$$

which we could write as

$$\frac{df_t}{f_t} = 3\,(\mu + \sigma^2)\,dt + 3\,\sigma\,dW_t.$$

7. Find solutions of the Black-Scholes terminal value problem

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2\,S^2\,\frac{\partial^2 V}{\partial S^2} + (r - y)\,S\,\frac{\partial V}{\partial S} - r\,V = 0, \quad S > 0,\ t < T,$$
$$V(S, T) = f(S), \quad S > 0,$$

when

(a) $f(S) = C$, where C is a constant;
In this case it makes sense to look for solutions which don't depend on S, so $\partial V/\partial S = 0$ and $\partial^2 V/\partial S^2 = 0$ identically. Then we have $V = V(t)$ and the partial differential equation reduces to the ODE

$$\frac{dV}{dt} - r\,V = 0, \quad V(T) = C$$

which has the solution $V = C\,e^{-r(T-t)}$.

(b) $f(S) = S^\alpha$, where α is a constant.
[Hint: you don't need the Feynman-Kăc formula to do either of these. Look for simple functional forms of the solution.]

In this case, let's look for separable solutions

$$V(S,t) = S^{\alpha}\, f(t), \quad f(T) = 1.$$

We find that

$$\frac{\partial V}{\partial t} = S^{\alpha}\,\dot{f}(t), \quad S\,\frac{\partial V}{\partial S} = \alpha\, S^{\alpha}\, f(t), \quad S^2\,\frac{\partial^2 V}{\partial S^2} = \alpha(\alpha-1)\, S^{\alpha}\, f(t)$$

and the Black-Scholes equation becomes

$$S^{\alpha}\left(\dot{f}(t) + \tfrac{1}{2}\sigma^2\ \alpha(\alpha-1)\, f(t) + (r-y)\,\alpha\, f(t) - r\, f(t)\right) = 0.$$

As this has to hold for all $S > 0$ we have an ordinary differential equation for $f(t)$ which we can write as

$$\dot{f}(t) = \lambda\, f(t), \quad f(T) = 1,$$

where

$$\lambda = r - (r-y)\,\alpha - \tfrac{1}{2}\sigma^2\,\alpha(\alpha-1).$$

The solution is

$$f(t) = e^{-\lambda(T-t)}$$

and

$$V(S,t) = e^{-\lambda(T-t)}\, S^{\alpha}.$$

Optional questions

8. The Black-Scholes equation from a binomial method.

   One step of the Cox, Ross & Rubinstein binomial method can be written as

$$V(S,t) = e^{-r\delta t}\Big(q\,V^u + (1-q)\,V^d\Big)$$

   where

$$V^u = V\big(S^u, t+\delta t\big), \quad V^d = V\big(S^d, t+\delta t\big),$$
$$S^u = S\,e^{\sigma\sqrt{\delta t}}, \quad S^d = S\,e^{-\sigma\sqrt{\delta t}}, \quad q = \frac{e^{r\delta t} - e^{-\sigma\sqrt{\delta t}}}{e^{\sigma\sqrt{\delta t}} - e^{-\sigma\sqrt{\delta t}}},$$

   $\sigma > 0$ is the volatility, r is the risk-free rate and $\delta t > 0$ is the length of the time-step. Supposing this is true for all $S > 0$ and that $V(S,t)$ may be expanded in a Taylor series in both S and t, show that as $\delta t \to 0$

$$\begin{aligned}
q &= \frac{1}{2} + \frac{r - \frac{1}{2}\sigma^2}{2\,\sigma}\sqrt{\delta t} + \mathcal{O}(\delta t), \\
V^u &= V + \sqrt{\delta t}\,\sigma\,S\,\frac{\partial V}{\partial S} + \delta t\left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2\Big(S\,\frac{\partial V}{\partial S} + S^2\,\frac{\partial^2 V}{\partial S^2}\Big)\right) + \mathcal{O}(\delta t^{3/2}), \\
V^d &= V - \sqrt{\delta t}\,\sigma\,S\,\frac{\partial V}{\partial S} + \delta t\left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2\Big(S\,\frac{\partial V}{\partial S} + S^2\,\frac{\partial^2 V}{\partial S^2}\Big)\right) + \mathcal{O}(\delta t^{3/2}),
\end{aligned}$$

   where V and all its partial derivatives are evaluated at (S,t). Hence show that in the limit $\delta t \to 0$ the option price satisfies the (zero dividend-yield) Black-Scholes equation

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2\,S^2\,\frac{\partial^2 V}{\partial S^2} + r\,S\,\frac{\partial S}{\partial S} - r\,V = 0.$$

   As $\delta t \to 0$ we can expand

$$\begin{aligned}
e^{\sigma\sqrt{\delta t}} &= 1 + \sigma\sqrt{\delta t} + \tfrac{1}{2}\sigma^2\delta t + \mathcal{O}(\delta t^{3/2}), \\
e^{-\sigma\sqrt{\delta t}} &= 1 - \sigma\sqrt{\delta t} + \tfrac{1}{2}\sigma^2\delta t + \mathcal{O}(\delta t^{3/2}), \\
e^{r\delta t} &= 1 + r\delta t + \mathcal{O}(\delta t^2),
\end{aligned}$$

   from which it follows that

$$\begin{aligned}
q &= \frac{\sigma\sqrt{\delta t} + (r - \frac{1}{2}\sigma^2)\delta t + \mathcal{O}(\delta t^{3/2})}{2\sigma\sqrt{\delta t} + \mathcal{O}(\delta t^{3/2})} \\
&= \frac{1}{2} + \frac{r - \frac{1}{2}\sigma^2}{2\sigma}\sqrt{\delta t} + \mathcal{O}(\delta t),
\end{aligned}$$

and that

$$
\begin{aligned}
&V(S\,e^{\sigma\sqrt{\delta t}}, t+\delta t) \\
&\qquad = V(S,t) + \delta t\,\frac{\partial V}{\partial t}(S,t) + \left(\sigma\sqrt{\delta t} + \tfrac{1}{2}\sigma^2\delta t\right) S\,\frac{\partial V}{\partial S}(S,t) \\
&\qquad\qquad + \tfrac{1}{2}\sigma^2\delta t\,S^2\,\frac{\partial^2 V}{\partial S^2}(S,t) + \mathcal{O}(\delta t^{3/2}) \\
&\qquad = V + \sqrt{\delta t}\,\sigma\,S\,\frac{\partial V}{\partial S} + \delta t\left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2\left(S\frac{\partial V}{\partial S} + S^2\frac{\partial^2 V}{\partial S^2}\right)\right) + \mathcal{O}(\delta t^{3/2}) \\
&V(S\,e^{-\sigma\sqrt{\delta t}}, t+\delta t) \\
&\qquad = V - \sqrt{\delta t}\,\sigma\,S\,\frac{\partial V}{\partial S} + \delta t\left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2\left(S\frac{\partial V}{\partial S} + S^2\frac{\partial^2 V}{\partial S^2}\right)\right) + \mathcal{O}(\delta t^{3/2}).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&q\,V(S\,e^{\sigma\sqrt{\delta t}}, t+\delta t) + (1-q)\,V(S\,e^{-\sigma\sqrt{\delta t}}, t+\delta t) \\
&\qquad = V + \delta t\left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2\,\frac{\partial^2 V}{\partial S^2} + \tfrac{1}{2}\sigma^2 S\,\frac{\partial V}{\partial S}\right) + (r - \tfrac{1}{2}\sigma^2)\,\delta t\,S\,\frac{\partial V}{\partial S} + \mathcal{O}(\delta t^{3/2}) \\
&\qquad = V + \delta t\left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2\,S^2\frac{\partial^2 V}{\partial S^2} + r\,S\,\frac{\partial V}{\partial S}\right) + \mathcal{O}(\delta t^{3/2})
\end{aligned}
$$

and hence

$$
\begin{aligned}
&e^{-r\delta t}\left(q\,V(S\,e^{\sigma\sqrt{\delta t}}, t+\delta t) + (1-q)\,V(S\,e^{-\sigma\sqrt{\delta t}}, t+\delta t)\right) \\
&\qquad = (1 - r\delta t)\left(V + \delta t\left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2\,S^2\frac{\partial^2 V}{\partial S^2} + r\,S\,\frac{\partial V}{\partial S}\right)\right) + \mathcal{O}(\delta t^{3/2}) \\
&\qquad = V + \delta t\left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2\,S^2\frac{\partial^2 V}{\partial S^2} + r\,S\,\frac{\partial V}{\partial S} - r\,V\right) + \mathcal{O}(\delta t^{3/2}).
\end{aligned}
$$

Substituting this back into the binomial equation gives

$$
V = V + \delta t\left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2\,S^2\frac{\partial^2 V}{\partial S^2} + r\,S\,\frac{\partial V}{\partial S} - r\,V\right) + \mathcal{O}(\delta t^{3/2})
$$

and cancelling V and dividing by δt then gives

$$
\left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2\,S^2\frac{\partial^2 V}{\partial S^2} + r\,S\,\frac{\partial V}{\partial S} - r\,V\right) + \mathcal{O}(\delta t^{1/2}) = 0.
$$

Taking the limit $\delta t \to 0$ gives the Black-Scholes equation,

$$
\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2\,S^2\frac{\partial^2 V}{\partial S^2} + r\,S\,\frac{\partial V}{\partial S} - r\,V = 0.
$$

9. The variation of a function, or stochastic process, over $[0, t]$, is

$$\langle f \rangle_t = \lim_{|\pi| \to 0} \sum_{k=0}^{n-1} |\, f_{k+1} - f_k \,|.$$

If $\langle f \rangle_t$ is finite on $[0, t]$ we say f has bounded variation on $[0, t]$. Show that:

(a) if f is $C^1[0, t]$ then $\langle f \rangle_t = \int_0^t |f'(t)|\, dt < \infty$;

If f is C^1 on $[0, t]$ then the intermediate value theorem asserts that $f_{k+1} - f_k = f'(\xi_k)(t_{k+1} - t_k)$ for some $\xi_k \in [t_k, t_{k+1}]$. Since $\delta t_k = (t_{k+1} - t_k) > 0$,

$$\langle f \rangle_t = \lim_{|\pi| \to 0} \sum_{k=0}^{n-1} |\, f'(\xi_k) \,|\, \delta t_k = \int_0^t |\, f'(s) \,|\, ds.$$

If $f'(s)$ is continuous on $[0, t]$ then so too is $|f'(s)|$ and as $[0, t]$ is a closed and bounded set, $|f'(s)|$ must be bounded on $[0, t]$. Therefore the integral must be finite and hence f has bounded variation on $[0, t]$.

(b) if f is a continuous function with $\langle f \rangle_t < \infty$ then its quadratic variation is zero, $[\, f \,]_t = 0$;

We have

$$[\, f \,]_t = \lim_{|\pi| \to 0} \sum_{k=0}^{n-1} \left(f_{k+1} - f_k \right)^2 \geq 0$$

and we also have

$$\sum_{k=0}^{n-1} \left(f_{k+1} - f_k \right)^2 \leq \left(\max_j |\, f_{j+1} - f_j \,| \right) \left(\sum_{k=0}^{n-1} |\, f_{k+1} - f_k \,| \right).$$

If f has bounded variation then the sum on the right-hand side is finite as $|\pi| \to 0$. If f is continuous then the maximum value of $|\, f_{j+1} - f_j \,| \to 0$ as $|\pi| \to 0$. Thus the product of the two vanishes as $|\pi| \to 0$.

(c) Brownian motion does not have bounded variation;

Brownian motion is continuous in t and has non-zero quadratic variation. Therefore it must have unbounded variation by the previous part of the question.

(d) the arc length of the graph of a Brownian motion is infinite for any $t > 0$.

[Hint: if $y = X_t$ has an arc length s then $ds = \sqrt{dy^2 + dt^2} \geq \sqrt{dy^2} = |dy|$.]

Assume that we can define arc length for a Brownian motion. Then it is given by

$$s = \int_0^t ds$$

where if $y = X_t$ then ds is given by

$$ds^2 = dt^2 + dy^2, \quad ds = \sqrt{dt^2 + dy^2}.$$

Now observe that

$$ds = \sqrt{dt^2 + dy^2} \geq \sqrt{dy^2} = |\, dy \,|.$$

If the integral of $|\, dy \,|$ existed it would equal the variation of W_t, but we know that W_t doesn't have bounded variation. This implies that the length of the graph of a Brownian motion is infinite (for any $t > 0$).