B8.3 Mathematical Models for Financial Derivatives

Hilary Term 2019

Problem Sheet Three

Your grade will be determined from the *best five* answers to the first *seven* questions.

1. Assume a zero interest rate, $r = 0$, in this problem (to avoid problems with the time-value of cash payments). Let $0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = t$ be a partition of the interval $[0, t]$. Let $S_u > 0$ be the price of a share at time $u \in [0, t]$, Δ_u be a number of shares at time u and abbreviate $S_{t_k} = S_k$, $\Delta_{t_k} = \Delta_k$. At time $t_0 = 0$ we buy Δ_0 shares, at price S_0, and hold these until time t_1. At time t_1 we buy (or sell) enough shares, at price S_1, so that we have Δ_1 shares, which we hold until time t_2, at which point we buy (or sell) enough shares, at price S_2, so that we have Δ_2 shares. We continue this process until time t_{n-1}, when we end up with Δ_{n-1} shares which we hold until $t_n = t$ at which point we sell all shares we hold, at price S_n. Show that the cost of this procedure is
$$-\sum_{j=0}^{n-1} \Delta_j \,(S_{j+1} - S_j).$$
[Hint: at time step t_{k+1} the change from holding Δ_k shares to holding Δ_{k+1} shares is equivalent to selling all the Δ_k shares and then buying back Δ_{k+1} shares, with both the trades being executed at share price S_{k+1}.]

   Following the hint, the cost is
$$\begin{aligned}\Delta_0\, S_0 \;&+\; (\Delta_1\, S_1 - \Delta_0\, S_1)\\ &+\; (\Delta_2\, S_2 - \Delta_1\, S_2)\\ &+\; \cdots\\ &+\; (\Delta_{n-1}\, S_{n-1} - \Delta_{n-2}\, S_{n-1})\\ &-\; \Delta_{n-1}\, S_n\end{aligned}$$
   which can be rearranged to read
$$\Delta_0(S_0 - S_1) + \Delta_1(S_1 - S_2) + \cdots \Delta_{n-2}(S_{n-2} - S_{n-1}) + \Delta_{n-1}(S_{n-1} - S_n)$$
   which can be written more briefly as
$$-\sum_{j=0}^{n-1} \Delta_j (S_{j+1} - S_j).$$

Show that, formally at least, in the limit $|\pi| \to 0$ the cost becomes

$$C_t = -\int_0^t \Delta_u \, dS_u$$

where the integral is an Itô integral (with respect to S_u) and hence deduce that

$$dC_t = -\Delta_t \, dS_t.$$

Formally we have

$$\lim_{|\pi|\to 0} \sum_{j=0}^{n-1} \Delta_j (S_{j+1} - S_j) = \int_0^t \Delta_u \, dS_u$$

by analogy with the Itô integral. Putting the $-$ sign back we get

$$C_t = -\int_0^t \Delta_u \, dS_u$$

and differentiating gives

$$dC_t = -\Delta_t \, dS_t.$$

2. Show that if $V(S,t)$ is a solution of the Black-Scholes equation (for $S > 0$ and $t < T$) then so too are:

   (a) $a\,V(S,t)$ with $a \in \mathbb{R}$;
   Linearity of the Black-Scholes equation.

   (b) $V(bS,t)$ with $b > 0$;
   Set $\hat{S} = b\,S > 0$ so that the chain rule gives

$$\frac{\partial}{\partial S} = \frac{\partial \hat{S}}{\partial S} \frac{\partial}{\partial \hat{S}} = b \frac{\partial}{\partial \hat{S}}.$$

   Multiplying this by S gives

$$S \frac{\partial}{\partial S} = b\,S \frac{\partial}{\partial \hat{S}} = \hat{S} \frac{\partial}{\partial \hat{S}}. \tag{1}$$

   From this we find that

$$S \frac{\partial}{\partial S}\left(S \frac{\partial}{\partial S}\right) = \hat{S} \frac{\partial}{\partial \hat{S}}\left(\hat{S} \frac{\partial}{\partial \hat{S}}\right)$$

   which, when expanded, reads

$$S^2 \frac{\partial^2}{\partial S^2} + S \frac{\partial}{\partial S} = \hat{S}^2 \frac{\partial^2}{\partial \hat{S}^2} + \hat{S} \frac{\partial}{\partial \hat{S}}.$$

Subtracting (1) then gives

$$S^2 \frac{\partial^2}{\partial S^2} = \hat{S}^2 \frac{\partial^2}{\partial \hat{S}^2}.$$

Thus, if for $S > 0$ we have

$$\frac{\partial V}{\partial t}(S,t) + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}(S,t) + (r-y)\, S\, \frac{\partial V}{\partial S}(S,t) - r\, V(S,t) = 0,$$

and we set $\hat{V}(\hat{S},t) = V(bS,t)$ then for $\hat{S} = bS > 0$ we also have

$$\frac{\partial \hat{V}}{\partial t}(\hat{S},t) + \tfrac{1}{2}\sigma^2 \hat{S}^2 \frac{\partial^2 \hat{V}}{\partial \hat{S}^2}(\hat{S},t) + (r-y)\, \hat{S}\, \frac{\partial \hat{V}}{\partial \hat{S}}(\hat{S},t) - r\, \hat{V}(\hat{S},t) = 0,$$

(c) $a\, V(bS,t)$ with $a \in \mathbb{R}$, $b > 0$.
Parts (a) and (b) together with linearity of the Black-Scholes equation.

3. A log-option is an option with the payoff function

$$P_{\mathrm{o}}(S_T) = \log(S_T/K),$$

where the "strike" is positive, $K > 0$. Find the Black-Scholes value function for a European log-option. (Such options are not traded, but they occur in the theory underlying the CBOE's VIX (variance index) which measures the S&P500 index's variance, allowing futures and options to be written on this variance.)

There are at least two methods to do this.

Method I: reduction to a system of ODEs

Observe that

$$\frac{\partial}{\partial S}\log(S/K) = \frac{1}{S}, \quad \frac{\partial^2}{\partial S^2}\log(S/K) = -\frac{1}{S^2},$$

which implies that

$$S\,\frac{\partial}{\partial S}\log(S/K) = 1, \quad S^2\,\frac{\partial^2}{\partial S^2}\log(S/K) = -1.$$

This suggests we try a solution of the form

$$V(S,t) = a(t)\,\log(S/K) + b(t),$$

which gives

$$\frac{\partial V}{\partial t} = \dot{a}(t)\log(S/K) + \dot{b}(t), \quad S\,\frac{\partial V}{\partial S} = a(t), \quad S^2\,\frac{\partial^2 V}{\partial S^2} = -a(t).$$

The payoff $V(S,T) = \log(S/K)$ gives the boundary conditions

$$a(T) = 1, \quad b(T) = 0.$$

Substituting into the Black-Scholes equation gives

$$\big(\dot{a}(t) - r\,a(t)\big)\log(S/K) + \Big(\dot{b}(t) - r\,b(t) + (r - y - \tfrac{1}{2}\sigma^2)\,a(t)\Big) = 0.$$

As this has to hold for all $S > 0$, we must have the IVPs

$$\begin{aligned} \dot{a}(t) - r\,a(t) &= 0, & a(T) &= 1, \\ \dot{b}(t) - r\,b(t) &= -(r - y - \tfrac{1}{2}\sigma^2)\,a(t), & b(T) &= 0. \end{aligned}$$

The first IVP integrates to give

$$a(t) = e^{-r(T-t)}$$

and, with the aid of an integrating factor, the second one then becomes

$$\frac{d}{dt}\Big(\,e^{r(T-t)}\,b(t)\,\Big) = -(r - y - \tfrac{1}{2}\sigma^2), \quad b(T) = 0,$$

which, when integrated from t to T, gives

$$b(T) - e^{r(T-t)}\,b(t) = -\int_t^T \big(r - y - \tfrac{1}{2}\sigma^2\big)\,du = \big(r - y - \tfrac{1}{2}\sigma^2\big)\,(T-t).$$

Thus

$$b(t) = e^{-r(T-t)}\big(r - y - \tfrac{1}{2}\sigma^2\big)\,(T-t)$$

and so

$$V(S,t) = e^{-r(T-t)}\,\Big(\log(S/K) + \big(r - y - \tfrac{1}{2}\sigma^2\big)\,(T-t)\Big)\,.$$

Method II: use the Feynman-Kăc representation

Write the solution as

$$V(S,t) = e^{-r(T-t)}\,\mathbb{E}_t\Big[\log\big(S_T/K\big)\,|\,S_t = S\Big],$$

where S_t evolves as

$$\frac{dS_t}{S_t} = (r - y)\,dt + \sigma\,dW_t.$$

Setting $\tau = T - t$, $S_t = S$ and integrating the SDE gives

$$S_T = S\,\exp\big((r - y - \tfrac{1}{2}\sigma^2)\,\tau + \sigma\,W_\tau\big)$$

from which it follows that

$$\log\big(S_T/K\big) = \log(S/K) + (r - y - \tfrac{1}{2}\sigma^2)\,\tau + \sigma\,W_\tau.$$

Taking expectations gives

$$\begin{aligned}
\mathbb{E}_t\Big[\log\big(S_T/K\big)\,|\,S_t = S\Big] &= \mathbb{E}_t\Big[\log(S/K) + (r - y - \tfrac{1}{2}\sigma^2)\,\tau + \sigma\,W_\tau\Big] \\
&= \log(S/K) + (r - y - \tfrac{1}{2}\sigma^2)\,\tau + \sigma\,\mathbb{E}_t\big[\,W_\tau\,\big] \\
&= \log(S/K) + (r - y - \tfrac{1}{2}\sigma^2)\,(T - t).
\end{aligned}$$

Therefore

$$V(S,t) = e^{-r(T-t)}\left(\log(S/K) + (r - y - \tfrac{1}{2}\sigma^2)\,(T - t)\right).$$

4. Find the Black-Scholes price function of a European digital call option, i.e., an option whose payoff function is

$$f(S_T) = \mathbf{1}_{\{S_T \ge K\}} = \begin{cases} 0 & \text{if } 0 < S_T < K, \\ 1 & \text{if } S_T \ge K. \end{cases}$$

There are numerous ways to do this. Here are three.

Method I: use the Feynman-Kăc representation

The Feynman-Kăc representation of the solution is

$$C_{\mathrm{d}}(S,t) = e^{-r(T-t)}\,\mathbb{E}_t\big[\,\mathbf{1}_{\{S_T \ge K\}}\,|\,S_t = S\,\big]$$

where S_t evolves as

$$\frac{dS_t}{S_t} = (r - y)\,dt + \sigma\,dW_t.$$

Note that $\mathbb{E}_t\big[\,\mathbf{1}_{\{S_T \ge K\}}\,\big]$ is also the probability that $S_T \ge K$.
Integrating the SDE from t to T with $\tau = T - t$ and $S_t = S$ gives

$$S_T = S\exp\left((r - y - \tfrac{1}{2}\sigma^2)\,\tau + \sigma\,W_\tau\right)$$

and so we can write

$$\begin{aligned}
\mathbb{E}_t\big[\,\mathbf{1}_{\{S_T \ge K\}}\,|\,S_t = S\,\big] &= \mathrm{prob}\big(\,S_T \ge K\,\big) \\
&= \mathrm{prob}\Big(\,S\exp\big(\,(r - y - \tfrac{1}{2}\sigma^2)\tau + \sigma\,W_\tau\,\big) \ge K\,\Big) \\
&= \mathrm{prob}\Big(\,\log S + (r - y - \tfrac{1}{2}\sigma^2)\tau + \sigma\,W_\tau \ge \log K\,\Big) \\
&= \mathrm{prob}\Big(\,\sigma\,W_\tau \ge \log(K/S) - (r - y - \tfrac{1}{2}\sigma^2)\tau\,\Big) \\
&= \mathrm{prob}\Big(\,\sqrt{\sigma^2\tau}\,Z \ge \log(K/S) - (r - y - \tfrac{1}{2}\sigma^2)\tau\,\Big),
\end{aligned}$$

where $Z \sim N(0,1)$. Therefore we have

$$\mathbb{E}_t\big[\,\mathbf{1}_{\{S_T \geq K\}}\,|\,S_t = S\,\big] = \text{prob}\Big(\,Z \geq \frac{\log(K/S) - (r - y - \frac{1}{2}\sigma^2)\tau}{\sqrt{\sigma^2\tau}}\,\Big)$$

This is the same thing as

$$\mathbb{E}_t\big[\,\mathbf{1}_{\{S_T \geq K\}}\,|\,S_t = S\,\big] = \text{prob}\Big(\,-Z \leq \frac{\log(S/K) + (r - y - \frac{1}{2}\sigma^2)\tau}{\sqrt{\sigma^2\tau}}\,\Big)$$

and since if $Z \sim N(0,1)$ then $-Z \sim N(0,1)$ (normal distributions are invariant under reflection about the mean, which in this case is zero) we see that

$$\begin{aligned}\mathbb{E}_t\big[\,\mathbf{1}_{\{S_T \geq K\}}\,|\,S_t = S\,\big] &= \text{prob}\Big(\,Z \leq \frac{\log(S/K) + (r - y - \frac{1}{2}\sigma^2)\tau}{\sqrt{\sigma^2\tau}}\,\Big) \\ &= \text{prob}\big(\,Z \leq d_-\,\big) \\ &= \mathrm{N}(d_-)\end{aligned}$$

where, as usual,

$$d_- = \frac{\log(S/K) + (r - y - \frac{1}{2}\sigma^2)(T-t)}{\sqrt{\sigma^2(T-t)}}.$$

Multiply this by $e^{-r(T-t)}$ gives the option price as

$$C_{\text{d}}(S,t) = e^{-r(T-t)}\,\mathrm{N}(d_-).$$

Method II: differentiate minus a call with respect to strike

The Black-Scholes price of a standard call option is

$$C_{\text{bs}}(S,t) = S\,e^{-y(T-t)}\,\mathrm{N}(d_+) - K\,e^{-r(T-t)}\,\mathrm{N}(d_-),$$

where as usual

$$d_\pm = \frac{\log(S/K) + (r - y \pm \frac{1}{2}\sigma^2)(T-t)}{\sqrt{\sigma^2(T-t)}}.$$

It follows that

$$d_+ - d_- = \sqrt{\sigma^2(T-t)}, \quad d_+ + d_- = \frac{2\log(S/K) + 2(r-y)(T-t)}{\sqrt{\sigma^2(T-t)}}$$

and

$$\tfrac{1}{2}\big(\,d_+^2 - d_-^2\,\big) = \log(S/K) + (r-y)(T-t)$$

or, equivalently,

$$S\,e^{-y(T-t)}\,e^{-\frac{1}{2}d_+^2} = K\,e^{-r(T-t)}\,e^{-\frac{1}{2}d_-^2}.$$

Note that $d_+ - d_- = \sqrt{\sigma^2(T-t)}$ implies that $\dfrac{\partial d_+}{\partial K} = \dfrac{\partial d_-}{\partial K}$.

From these it follows that

$$\begin{aligned}
\frac{\partial C_{\mathrm{bs}}}{\partial K}(S,t) &= -e^{-r(T-t)}\,\mathrm{N}(d_-) \\
&+ S\,e^{-y(T-t)}\,\mathrm{N}'(d_+)\,\frac{\partial d_+}{\partial K} - K\,e^{-r(T-t)}\,\mathrm{N}'(d_-)\,\frac{\partial d_-}{\partial K} \\
&= -e^{-r(T-t)}\,\mathrm{N}(d_-) \\
&+ \frac{1}{\sqrt{2\pi}}\left(S\,e^{-y(T-t)}\,e^{-\frac{1}{2}d_+^2} - K\,e^{-r(T-t)}\,e^{-\frac{1}{2}d_-^2}\right)\frac{\partial d_+}{\partial K} \\
&= -e^{-r(T-t)}\,\mathrm{N}(d_-).
\end{aligned}$$

Differentiating the payoff, $C_{\mathrm{bs}}(S,T) = \max(S-K,0)$, gives

$$\frac{\partial C_{\mathrm{bs}}}{\partial K}(S,T) = \begin{cases} 0 & \text{if } 0 < S < K \\ -1 & \text{if } 0 < K < S, \end{cases}$$

which is minus the digital call's payoff.

Note that K does not explicitly occur in the Black-Scholes equation so we find that

$$\frac{\partial}{\partial K}\left(\frac{\partial C_{\mathrm{bs}}}{\partial t} + \tfrac{1}{2}\sigma^2\,S^2\,\frac{\partial^2 C_{\mathrm{bs}}}{\partial S^2} + (r-y)\,S\,\frac{\partial C_{\mathrm{bs}}}{\partial S} - r\,C_{\mathrm{bs}}\right) = 0$$

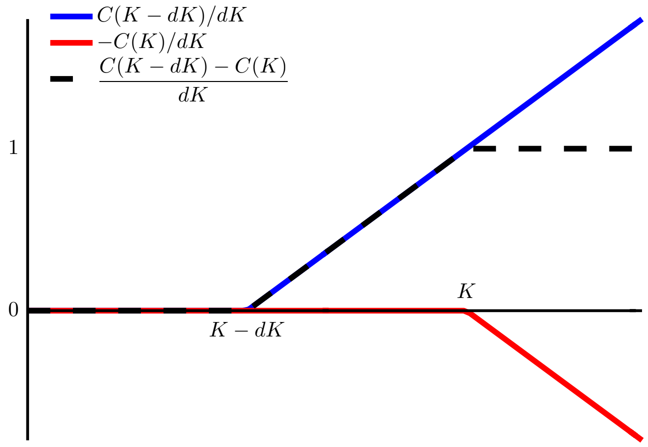
and hence, swapping the order of differentiation ($\partial C_{\mathrm{bs}}/\partial K$ is clearly sufficiently differentiable for $t < T$)

$$\frac{\partial}{\partial t}\left(\frac{\partial C_{\mathrm{bs}}}{\partial K}\right)+\tfrac{1}{2}\sigma^2\,S^2\,\frac{\partial^2}{\partial S^2}\left(\frac{\partial C_{\mathrm{bs}}}{\partial K}\right)+(r-y)\,S\,\frac{\partial}{\partial S}\left(\frac{\partial C_{\mathrm{bs}}}{\partial K}\right)-r\left(\frac{\partial C_{\mathrm{bs}}}{\partial K}\right) = 0.$$

This shows that $\partial C_{\mathrm{bs}}/\partial K$ is a solution of the Black-Scholes equation (for $t < T$). In order to get the payoff of the digital call we simply take $-\partial C_{\mathrm{bs}}/\partial K$. That is, the function

$$C_{\mathrm{d}}(S,t) = -\frac{\partial C_{\mathrm{bs}}}{\partial S}(S,t) = e^{-r(T-t)}\,\mathrm{N}(d_-)$$

satisfies the Black-Scholes problem for the digital call option.



Method III: market practice

Typically what a trader will do to synthesise a digital call is first go long a call with strike $K - dK$ and short a call with strike K, both with the same expiry T. The payoff is

$$C_{\rm bs}(S,T;K-dK)-C_{\rm bs}(S,T;K) = \begin{cases} 0 & \text{if } S \le K-dK, \\ S-K+dK & \text{if } K-dK < S < K, \\ dK & \text{if } S \ge K. \end{cases}$$

To scale this up so the payoff varies between 0 and 1 (rather than 0 and dK), they take $1/dK$ in each of the calls to get the continuous piecewise linear function

$$\frac{C_{\rm bs}(S,T;K-dK)-C_{\rm bs}(S,T;K)}{dK} = \begin{cases} 0 & \text{if } S \le K-dK, \\ 1+\frac{(S-K)}{dK} & \text{if } K-dK < S < K, \\ 1 & \text{if } S \ge K. \end{cases}$$

They then choose dK as small as possible. (If $K - dK < S_T < K$ they make a profit of $1 + (S_T - K)/dK$ so this is an arbitrage—this is one way traders make their money.)

The value of the position before expiry is simply

$$\frac{C(S,t;K-dK)-C(S,t;K)}{dK}$$

and in the limit that $dK \to 0$ this gives

$$\lim_{dK\to 0}\frac{C(S,t;K-dK)-C(S,t;K)}{dK} = -\frac{\partial C}{\partial K}(S,t;K)$$

as above.

A European digital put option has the payoff $f(S_T) = \mathbf{1}_{\{S_T<K\}}$. Use a no arbitrage argument to establish a digital put-call parity result and hence find the Black-Scholes price function for a digital put.

Let $C_{\mathrm{d}}(S,t)$ be the price of the digital call and $P_{\mathrm{d}}(S,t)$ be the price of the digital put, both having the same strike $K > 0$ and expiry date T. At expiry we either have

$$0 < S_T < K \quad \text{or} \quad 0 < K \leq S_T.$$

The two events are mutually exclusive and one or the other must happen. In the first case, $C_{\mathrm{d}}(S_T,T) = 0$ and $P_{\mathrm{d}}(S_T,T) = 1$. In the second case $C_{\mathrm{d}}(S_T,T) = 1$ and $P_{\mathrm{d}}(S_T,T) = 0$. Thus we always have

$$C_{\mathrm{d}}(S_T,T) + P_{\mathrm{d}}(S_T,T) = 1.$$

That is, if we hold both we are guaranteed one unit of currency at time T. The present value of one unit of currency at T at some time $t < T$ is $e^{-r(T-t)}$ and so digital put-call parity is

$$C_{\mathrm{d}}(S_T,t) + P_{\mathrm{d}}(S_T,t) = e^{-r(T-t)}.$$

Therefore we have

$$\begin{aligned} P_{\mathrm{d}}(S,t) &= e^{-r(T-t)} - C_{\mathrm{d}}(S,t) \\ &= e^{-r(T-t)} - e^{-r(T-t)}\,\mathrm{N}(d_-) \\ &= e^{-r(T-t)}\Big(1 - \mathrm{N}(d_-)\Big) \\ &= e^{-r(T-t)}\,\mathrm{N}(-d_-). \end{aligned}$$

5. Show that if $V(S,t)$ is a sufficiently differentiable solution of the Black-Scholes equation (for $S > 0$ and $t < T$) then so too is

$$W(S,t) = S\,\frac{\partial V}{\partial S}(S,t).$$

We have

$$\frac{\partial W}{\partial t} = S\,\frac{\partial}{\partial S}\left(\frac{\partial V}{\partial t}\right), \quad S\,\frac{\partial W}{\partial S} = S\,\frac{\partial}{\partial S}\left(S\,\frac{\partial V}{\partial S}\right)$$

and

$$\begin{aligned} S^2\frac{\partial^2 W}{\partial S^2} &= 2S^2\frac{\partial^2 V}{\partial S^2} + S^3\frac{\partial^3 V}{\partial S^3} \\ &= S\left(2S\,\frac{\partial^2 V}{\partial S^2} + S^2\frac{\partial^3 V}{\partial S^3}\right) \\ &= S\,\frac{\partial}{\partial S}\left(S^2\frac{\partial^2 V}{\partial S^2}\right). \end{aligned}$$

Therefore, substituting W into the Black-Scholes equation gives

$$\begin{aligned}
&\frac{\partial W}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 W}{\partial S^2} + (r-y)\, S \frac{\partial W}{\partial S} - r\,W \\
&\quad = S\frac{\partial}{\partial S}\left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-y)\, S \frac{\partial V}{\partial S} - r\,V\right) \\
&\quad = S\frac{\partial}{\partial S}(0) \;=\; 0.
\end{aligned}$$

By induction, conclude that if $V(S,t)$ is sufficiently differentiable then

$$\left(S\frac{\partial}{\partial S}\right)^n V(S,t), \quad S^n \frac{\partial^n V}{\partial S^n}(S,t), \quad n = 2, 3, 4, \ldots$$

are also solutions of the Black-Scholes equation.

Let $V(S,t)$ satisfy the Black-Scholes equation and let

$$W_1 = S\frac{\partial}{\partial S}V,\ W_2 = \left(S\frac{\partial}{\partial S}\right)^2 V,\ W_3 = \left(S\frac{\partial}{\partial S}\right)^3 V, \ldots$$

We have just shown that (assuming sufficient differentiability) $W_1(S,t)$ is a solution of the Black-Scholes equation. It is clear that

$$W_2(S,t) = S\frac{\partial W_1}{\partial S}(S,t)$$

and so (assuming sufficient differentiability) $W_2(S,t)$ is also a solution. Suppose that $W_1(S,t)$ through to $W_m(S,t)$ are all solutions. Then we have

$$W_{m+1}(S,t) = S\frac{\partial W_m}{\partial S}(S,t)$$

which is, by the previous part of the question, also a solution (assuming enough differentiability). Thus if $V(S,t)$ is a solution so too is

$$\left(S\frac{\partial}{\partial S}\right)^n V(S,t)$$

for $n = 1,\ 2,\ 3, \ldots$ (assuming sufficient differentiability of $V(S,t)$).

Now observe that both of

$$S\frac{\partial V}{\partial S},$$

and

$$\left(S\frac{\partial}{\partial S}\right)^2 V(S,t) = S\frac{\partial}{\partial S}\left(S\frac{\partial V}{\partial S}\right) = S^2\frac{\partial^2 V}{\partial S^2} + S\frac{\partial V}{\partial S}$$

are both solutions of the Black-Scholes equation. Since the Black-Scholes equation is linear it follows that the difference of these two solutions,

$$S^2 \frac{\partial^2 V}{\partial S^2}$$

is a solution. Thus the second result is true for $n = 1$ and $n = 2$. Suppose it is true up to and including n, so

$$S \frac{\partial V}{\partial S}, \; S^2 \frac{\partial^2 V}{\partial S^2}, \ldots, \; S^n \frac{\partial^n V}{\partial S^n}$$

are all solutions. Then observe that

$$S \frac{\partial}{\partial S} \left(S^n \frac{\partial^n V}{\partial S^n} \right) = n \, S^n \frac{\partial^n V}{\partial S^n} + S^{n+1} \frac{\partial^{n+1} V}{\partial S^{n+1}}$$

but we know that

$$S \frac{\partial}{\partial S} \left(S^n \frac{\partial^n V}{\partial S^n} \right) \quad \text{and} \quad n \, S^n \frac{\partial^n V}{\partial S^n}$$

are solutions and so, by linearity,

$$S^{n+1} \frac{\partial^{n+1} V}{\partial S^{n+1}} = S \frac{\partial}{\partial S} \left(S^n \frac{\partial^n V}{\partial S^n} \right) - n \, S^n \frac{\partial^n V}{\partial S^n}$$

is also a solution.

See Question 8 for another, possibly simpler, way to do this question.

6. Let $C_{\text{bs}}(S, t; K, T, r, y, \sigma)$ denote the solution of a Black-Scholes call value problem with strike K, expiry date T, risk-free rate r, continuous dividend yield y and volatility σ. Consider the Black-Scholes problem

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - y)\, S \frac{\partial V}{\partial S} - r\, V = 0, \quad S > 0, \; t < T$$

$$V(S, T) = \frac{1}{K^2} \max\bigl(S^3 - K^3, 0\bigr), \quad S > 0.$$

Show that

$$V(S, t) = \frac{1}{K^2} C_{\text{bs}}(S^3, t; K^3, T, r, \hat{y}, \hat{\sigma})$$

where $\hat{y} = 3y - 2r - 3\sigma^2$ and $\hat{\sigma} = 3\sigma$.

[Hint: either write $\hat{S} = S^3$ and do a change of variables in the terminal value problem or think about the payoff and risk-neutral process for $\hat{S}_t = S_t^3$.]

As the hint suggests, there are (at least) two ways to do this question.

Method I: change of variables in the Black-Scholes equation

Put $\hat{S} = S^3$, $\hat{K} = K^3$ and $\hat{V}(\hat{S},t) = V(S,t)$ in the Black-Scholes problem. The chain rule gives

$$\frac{\partial}{\partial S} = \frac{\partial \hat{S}}{\partial S}\frac{\partial}{\partial \hat{S}} = 3\,S^2\,\frac{\partial}{\partial \hat{S}}$$

and so

$$S\,\frac{\partial}{\partial S} = 3\,\hat{S}\frac{\partial}{\partial \hat{S}}. \tag{2}$$

Applying this to itself we get

$$S\,\frac{\partial}{\partial S}\left(S\,\frac{\partial}{\partial S}\right) = 9\,\hat{S}\frac{\partial}{\partial \hat{S}}\left(\hat{S}\frac{\partial}{\partial \hat{S}}\right)$$

which gives

$$S^2\frac{\partial^2}{\partial S^2} + S\,\frac{\partial}{\partial S} = 9\,\hat{S}^2\frac{\partial^2}{\partial \hat{S}^2} + 9\,\hat{S}\frac{\partial}{\partial \hat{S}}$$

and subtracting (2) gives

$$S^2\,\frac{\partial^2}{\partial S^2} = 9\,\hat{S}^2\frac{\partial^2}{\partial \hat{S}^2} + 6\,\hat{S}\frac{\partial}{\partial \hat{S}}.$$

Thus we have

$$\frac{\partial V}{\partial t} = \frac{\partial \hat{V}}{\partial t}, \quad S\,\frac{\partial V}{\partial S} = 3\,\hat{S}\,\frac{\partial \hat{V}}{\partial \hat{S}}, \quad S^2\,\frac{\partial^2 V}{\partial S^2} = 9\,\hat{S}^2\,\frac{\partial^2 \hat{V}}{\partial \hat{S}^2} + 6\,\hat{S}\,\frac{\partial \hat{V}}{\partial \hat{S}}$$

and so, in terms of $\hat{V}$ and $\hat{S}$, the Black-Scholes equation becomes

$$\frac{\partial \hat{V}}{\partial t} + \tfrac{1}{2}\sigma^2\left(9\,\hat{S}^2\,\frac{\partial^2 \hat{V}}{\partial \hat{S}^2} + 6\,\hat{S}\,\frac{\partial \hat{V}}{\partial \hat{S}}\right) + 3(r-y)\,\hat{S}\,\frac{\partial \hat{V}}{\partial \hat{S}} - r\,\hat{V} = 0$$

or

$$\frac{\partial \hat{V}}{\partial t} + \tfrac{1}{2}(3\sigma)^2\hat{S}^2\,\frac{\partial^2 \hat{V}}{\partial \hat{S}^2} + 3(r-y+\sigma^2)\,\hat{S}\,\frac{\partial \hat{V}}{\partial \hat{S}} - r\,\hat{V} = 0.$$

Setting

$$\begin{aligned} \hat{\sigma} &= 3\,\sigma, & r - \hat{y} &= 3r - 3y + 3\sigma^2 \\ & & \hat{y} &= 3y - 2r - 3\sigma^2, \end{aligned}$$

we can write this as

$$\frac{\partial \hat{V}}{\partial t} + \tfrac{1}{2}\hat{\sigma}^2\hat{S}^2\,\frac{\partial^2 \hat{V}}{\partial \hat{S}^2} + (r-\hat{y})\,\hat{S}\,\frac{\partial \hat{V}}{\partial \hat{S}} - r\,\hat{V} = 0,$$

which is the standard Black-Scholes equation for $\hat{V}(\hat{S},t)$, but with $\hat{\sigma}$ and $\hat{y}$ replacing σ and y. The payoff becomes

$$\hat{V}(\hat{S},T) = \frac{1}{K^2}\,\max(\hat{S}-\hat{K},0).$$

Thus $\hat{V}(\hat{S},t)$ is $1/K^2$ regular calls, but with volatility $\hat{\sigma}$ and dividend yield $\hat{y}$, i.e.,

$$\hat{V}\big(\hat{S},t\big) = \frac{1}{K^2}\, C_{\rm bs}\big(\hat{S},t;\hat{K},T,r,\hat{y},\hat{\sigma}\big).$$

Reverting to the original variables, this gives

$$V(S,t) = \frac{1}{K^2}\, C_{\rm bs}\big(S^3,t;K^3,T,r,\hat{y},\hat{\sigma}\big),$$

with $\hat{\sigma} = 3\,\sigma$ and $\hat{y} = 3y - 2r - 3\sigma^2$.

Method II: change of variables in the Feynman-Kăc solution

In this case we go back to the Feynman-Kăc solution

$$V(S,t) = \frac{1}{K^2}\, e^{-r(T-t)}\, \mathbb{E}_t\big[\, (S_T^3 - K^3)^+ \,\big|\, S_t = S\,\big], \tag{3}$$

where S_t evolves as

$$\frac{dS_t}{S_t} = (r-y)\, dt + \sigma\, dW_t. \tag{4}$$

If S_t evolves according to (4) then Itô's lemma tells us that $\hat{S}_t = S_t^3$ evolves as

$$\begin{aligned} d\hat{S}_t &= 3\, S_t^2\, dS_t + 3\, S_t\, \sigma^2\, S_t^2\, dt \\ &= 3\,(r-y)\, S_t^3\, dt + 3\,\sigma\, S_t^3\, dW_t + 3\,\sigma^2\, S_t^3\, dt \\ &= 3\,\big(r - y + \sigma^2\big)\, \hat{S}_t\, dt + 3\,\sigma\, \hat{S}_t\, dW_t. \end{aligned}$$

If we set $\hat{\sigma} = 3\,\sigma$ and $r - \hat{y} = 3r - 3y + 3\sigma^2$, i.e., $\hat{y} = 3y - 2r - 3\sigma^2$, then we can write this as

$$\frac{d\hat{S}_t}{\hat{S}_t} = (r-\hat{y})\, dt + \hat{\sigma}\, dW_t. \tag{5}$$

Now if we set $\hat{K} = K^3$ and note that if $S_t = S$ then $\hat{S}_t = S^3$, we can write (3) in the form

$$V(S,t) = \frac{1}{K^2}\, e^{-r(T-t)}\, \mathbb{E}_t\big[\, (\hat{S}_T - \hat{K})^+ \,\big|\, \hat{S}_t = S^3\,\big],$$

where $\hat{S}_t$ evolves according to (5). This is precisely the formula for $1/K^2$ call options with initial price S^3, strike $\hat{K} = K^3$, expiry T, risk-free rate r, dividend yield $\hat{y} = 3y - 2r - 3\sigma^2$ and volatility $\hat{\sigma} = 3\sigma$. Thus

$$V(S,t) = \frac{1}{K^2}\, C_{\rm bs}\big(\, S^3,t;K^3,T,r,\hat{y},\hat{\sigma}\,\big)$$

7. Show that if $V(S,t)$ is a solution of the Black-Scholes equation (for $S > 0$ and $t < T$) and $B > 0$ then

$$W(S,t) = \left(\frac{S}{B}\right)^{2\alpha} V\left(\frac{B^2}{S}, t\right),$$

where $2\,\alpha = 1 - 2(r-y)/\sigma^2)$, is also a solution of the (same) Black-Scholes equation.

[Hint: put $\xi = B^2/S$ and note that $V(\xi,t)$ satisfies the Black-Scholes equation in $\xi > 0$ and $t < T$.]

As the hint suggests, write

$$W(S,t) = \left(\frac{S}{B}\right)^{\beta} V(\xi,t), \quad \xi = \frac{B^2}{S},$$

where $\beta = 2\,\alpha = 1 - 2(r-y)/\sigma^2$, and note that

$$\frac{\partial \xi}{\partial S} = -\frac{B^2}{S^2} = -\frac{\xi}{S}.$$

Clearly we have

$$\frac{\partial W}{\partial t}(S,t) = \left(\frac{S}{B}\right)^{\beta} \frac{\partial V}{\partial t}(\xi,t). \tag{6}$$

The chain rule gives

$$\begin{aligned}
\frac{\partial W}{\partial S}(S,t) &= \left(\frac{S}{B}\right)^{\beta}\left(\frac{\beta}{S} V(\xi,t) + \frac{\partial \xi}{\partial S}\frac{\partial V}{\partial \xi}(\xi,t)\right) \\
&= \frac{1}{S}\left(\frac{S}{B}\right)^{\beta}\left(\beta\, V(\xi,t) - \xi\,\frac{\partial V}{\partial \xi}(\xi,t)\right)
\end{aligned}$$

and so

$$S\,\frac{\partial W}{\partial S}(S,t) = \left(\frac{S}{B}\right)^{\beta}\left(\beta\, V(\xi,t) - \xi\,\frac{\partial V}{\partial \xi}(\xi,t)\right) \tag{7}$$

We then find that

$$\begin{aligned}
\frac{\partial^2 W}{\partial S^2}(S,t) &= \frac{\beta-1}{S^2}\left(\frac{S}{B}\right)^{\beta}\left(\beta\, V(\xi,t) - \xi\,\frac{\partial V}{\partial \xi}(\xi,t)\right) \\
&+ \frac{1}{S}\left(\frac{S}{B}\right)^{\beta}\frac{\partial \xi}{\partial S}\frac{\partial}{\partial \xi}\left(\beta\, V(\xi,t) - \xi\,\frac{\partial V}{\partial \xi}(\xi,t)\right) \\
&= \frac{1}{S^2}\left(\frac{S}{B}\right)^{\beta}\left\{\begin{array}{l}(\beta-1)\left(\beta\, V(\xi,t) - \xi\,\frac{\partial V}{\partial \xi}(\xi,t)\right) - \\ \xi\left((\beta-1)\,\frac{\partial V}{\partial \xi}(\xi,t) - \xi\,\frac{\partial^2 V}{\partial \xi^2}(\xi,t)\right)\end{array}\right\}
\end{aligned}$$

from which it follows that

$$
\begin{aligned}
&S^2 \, \frac{\partial^2 W}{\partial S^2}(S,t) \\
&\quad = \left(\frac{S}{B}\right)^{\beta} \left(\xi^2 \, \frac{\partial^2 V}{\partial \xi^2}(\xi,t) - 2(\beta-1)\,\xi\,\frac{\partial V}{\partial \xi}(\xi,t) + \beta(\beta-1)\,V(\xi,t) \right). \qquad (8)
\end{aligned}
$$

Substituting (6)–(8) into the Black-Scholes equation for $W(S,t)$ gives

$$
\begin{aligned}
&\frac{\partial W}{\partial t} + \tfrac{1}{2}\sigma^2\,S^2\,\frac{\partial^2 W}{\partial S^2} + (r-y)\,S\,\frac{\partial W}{\partial S} - r\,W \\
&\quad = \left(\frac{S}{B}\right)^{\beta} \left\{ \begin{array}{c} \dfrac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 \left(\xi^2\,\dfrac{\partial^2 V}{\partial \xi} - 2(\beta-1)\,\xi\,\dfrac{\partial V}{\partial \xi} + \beta(\beta-1)\,V \right) \\ + (r-y)\left(\beta\,V - \xi\,\dfrac{\partial V}{\partial \xi}\right) - r\,V \end{array} \right\} \\
&\quad = \left(\frac{S}{B}\right)^{\beta} \left\{ \begin{array}{c} \dfrac{\partial V}{\partial t}(\xi,t) + \tfrac{1}{2}\sigma^2\,\xi^2\,\dfrac{\partial^2 V}{\partial \xi^2}(\xi,t) \\ - \left(\sigma^2\,(\beta-1) + (r-y)\right)\xi\,\dfrac{\partial V}{\partial \xi}(\xi,t) \\ + \left(\beta\left(\tfrac{1}{2}\sigma^2\,(\beta-1) + (r-y)\right) - r\right) V(\xi,t) \end{array} \right\}
\end{aligned}
$$

Now observe that $\beta - 1 = -2(r-y)/\sigma^2$ so that

$$
-\left(\sigma^2(\beta-1) + (r-y)\right) = (r-y), \quad \tfrac{1}{2}\sigma^2(\beta-1) + (r-y) = 0
$$

which means that

$$
\begin{aligned}
&\frac{\partial W}{\partial t} + \tfrac{1}{2}\sigma^2\,S^2\,\frac{\partial^2 W}{\partial S^2} + (r-y)\,S\,\frac{\partial W}{\partial S} - r\,W \\
&\quad = \left(\frac{S}{B}\right)^{\beta} \left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2\,\xi^2\,\frac{\partial^2 V}{\partial \xi^2} + (r-y)\,\xi\,\frac{\partial V}{\partial \xi} - r\,V \right) \\
&\quad = 0.
\end{aligned}
$$

You might look at Question 9 to see one way to *derive* this result.

Optional questions

8. Let $V(S,t)$ satisfy the Black-Scholes problem

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-y)\,S\,\frac{\partial V}{\partial S} - r\,V = 0 \quad S>0,\ t<T,$$
$$V(S,t) = P_o(S), \quad S>0.$$

For some fixed reference price, $S_0 > 0$, set the dimensionless variables $x = \log(S/S_0)$, $\tau = \sigma^2(T-t)$ and $v(x,\tau) = V(S,t)/S_0$. Show that

$$\begin{aligned}\frac{\partial v}{\partial \tau} &= \tfrac{1}{2}\frac{\partial^2 v}{\partial x^2} + k_1\frac{\partial v}{\partial x} - k_2\,v, \quad x\in\mathbb{R},\ \tau>0,\\ v(x,0) &= p(x), \quad x\in\mathbb{R},\end{aligned} \tag{9}$$

where k_1 and k_2 are constants which you should find (in terms of r, y and σ) and $p(x)$ is a function which you should also find (in terms of $P_o(S)$).

With

$$x = \log(S/S_0), \quad \tau = \sigma^2(T-t), \quad v(x,\tau) = V(S,t)/S_0$$

we have

$$\frac{\partial}{\partial S} = \frac{\partial x}{\partial S}\frac{\partial}{\partial x} = \frac{1}{S}\frac{\partial}{\partial x}$$

and hence

$$S\frac{\partial}{\partial S} = \frac{\partial}{\partial x}.$$

Applying this formula to itself gives

$$\frac{\partial^2}{\partial x^2} = S\frac{\partial}{\partial S}\left(S\frac{\partial}{\partial S}\right) = S^2\frac{\partial^2}{\partial S^2} + S\frac{\partial}{\partial S}$$

and subtracting the previous equation gives

$$S^2\frac{\partial^2}{\partial S^2} = \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}.$$

Thus

$$S\frac{\partial V}{\partial S} = S_0\frac{\partial v}{\partial x}, \quad S^2\frac{\partial^2 V}{\partial S^2} = S_0\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\right).$$

We also have

$$\frac{\partial}{\partial t} = \frac{\partial \tau}{\partial t}\frac{\partial}{\partial \tau} = -\sigma^2\frac{\partial}{\partial \tau}$$

and so

$$\frac{\partial V}{\partial t} = -\sigma^2 S_0\frac{\partial v}{\partial \tau}.$$

Substituting these expressions into the Black-Scholes equation gives

$$-\sigma^2 S_0 \frac{\partial v}{\partial \tau} + \tfrac{1}{2}\sigma^2 S_0 \left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\right) + (r-y)\, S_0 \frac{\partial v}{\partial x} - r\, S_0\, v = 0$$

and cancelling the common factor of S_0, dividing by σ^2 and rearranging gives

$$\frac{\partial v}{\partial \tau} = \tfrac{1}{2} \frac{\partial^2 v}{\partial x^2} + \left(\frac{(r-y)}{\sigma^2} - \tfrac{1}{2}\right) \frac{\partial v}{\partial x} - \frac{r}{\sigma^2}\, v.$$

With the constants k_1 and k_2 defined as

$$k_1 = \frac{(r-y)}{\sigma^2} - \tfrac{1}{2}, \quad k_2 = \frac{r}{\sigma^2}$$

this reduces to

$$\frac{\partial v}{\partial \tau} = \tfrac{1}{2} \frac{\partial^2 v}{\partial x^2} + k_1 \frac{\partial v}{\partial x} - k_2\, v. \tag{10}$$

Clearly $t = T$ translates to $\tau = 0$ and so the terminal condition switches to an initial condition

$$S_0\, v(x, 0) = P_o(S) = P_o\Big(\, S_0\, e^x \Big)$$

and so

$$v(x, 0) = \frac{1}{S_0} P_o\Big(\, S_0\, e^x \Big) = p(x). \tag{11}$$

Assuming that $p(x)$ is a "reasonable" function[1], it can be shown that the solution of (9) is infinitely differentiable in x and τ for $\tau > 0$. Hence deduce that

$$v_n(x, \tau) = \frac{\partial^n v}{\partial x^n}(x, \tau), \quad n = 1, 2, 3, \ldots$$

are also solution of the partial differential equation in (9) for $\tau > 0$.

If v is infinitely differentiable in x and τ for $\tau > 0$ then we have

$$\frac{\partial^n}{\partial x^n} \frac{\partial v}{\partial \tau} = \frac{\partial}{\partial \tau} \frac{\partial^n v}{\partial x^n}, \quad \frac{\partial^n}{\partial x^n} \frac{\partial v}{\partial x} = \frac{\partial}{\partial x} \frac{\partial^n v}{\partial x^n}, \quad \frac{\partial^n}{\partial x^n} \frac{\partial^2 v}{\partial x^2} = \frac{\partial^2}{\partial x^2} \frac{\partial^n v}{\partial x^n},$$

and so taking $\partial^n / \partial x^n$ of the partial differential equation for v gives

$$\frac{\partial^n}{\partial x^n} \left(\frac{\partial v}{\partial \tau}\right) = \frac{\partial^n}{\partial x^n} \left(\tfrac{1}{2} \frac{\partial^2 v}{\partial x^2} + k_1 \frac{\partial v}{\partial x} - k_2\, v\right)$$

[1] For example, if $p(x)$ is integrable on every compact subset of $\mathbb{R}$ and there are constants $C > 0$ and $\kappa > 0$ with $|\, p(x)\,| < C\, e^{\kappa x^2}$ for all x ensures that the solution

$$u(x, \tau) = \frac{1}{\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} p(\xi)\, e^{-(x-\xi)^2/2\tau}\, d\xi$$

is C^∞ in x for $0 < \tau < 1/2\kappa$.

and hence interchanging the order of differentiation we have

$$\frac{\partial}{\partial \tau}\left(\frac{\partial^n v}{\partial x^n}\right) = \tfrac{1}{2}\frac{\partial^2}{\partial x^2}\left(\frac{\partial^n v}{\partial x^n}\right) + k_1\frac{\partial}{\partial x}\left(\frac{\partial^n v}{\partial x^n}\right) - k_2\frac{\partial^n v}{\partial x^n}$$

showing that

$$v_n(x,t) = \frac{\partial^n v}{\partial x^n}(x,t)$$

is also a solution of the partial differential equation (for $\tau > 0$).

Infer that if $P_o(S)$ is a "reasonable" function then

$$V_n(S,t) = \left(S\frac{\partial}{\partial S}\right)^n V(S,t), \quad n = 1,2,3,\ldots$$

are also solutions of the Black-Scholes partial differential equation for $t < T$.

Reasonable here means that $V(S,t)$ is infinitely differentiable in S and t for $t < T$. Since

$$\frac{\partial}{\partial x} = S\,\frac{\partial}{\partial S}$$

it follows that

$$S_0\,\frac{\partial^n}{\partial x^n}v(x,t) = \left(S\,\frac{\partial}{\partial S}\right)^n V(S,t)$$

and as the constant coefficient equation in x and τ is equivalent to the Black-Scholes equation in S and t, it follows that if $V(S,t)$ satisfies the Black-Scholes equation then

$$V_n(S,t) = \left(S\,\frac{\partial}{\partial S}\right)^n V(S,t)$$

also satisfies the Black-Scholes partial differential equation (for $t < T$).

9. Show that if we put

$$v(x,\tau) = e^{A\tau + Cx}\,u(x,\tau),$$

in (9) then, for certain values of A and C, which you should determine, we can reduce (9) to the heat equation problem

$$\begin{gathered}\frac{\partial u}{\partial \tau} = \tfrac{1}{2}\frac{\partial^2 u}{\partial x^2}, \quad x \in \mathbb{R},\ \tau > 0 \\ u(x,0) = q(x), \quad x \in \mathbb{R}.\end{gathered} \tag{12}$$

If we set

$$v(x,\tau) = e^{A\tau + Cx}\,u(x,\tau)$$

we find that

$$\begin{aligned}\frac{\partial v}{\partial \tau} &= e^{A\tau+Cx}\left(Au+\frac{\partial u}{\partial \tau}\right),\\ \frac{\partial v}{\partial x} &= e^{A\tau+Cx}\left(Cu+\frac{\partial u}{\partial x}\right),\\ \frac{\partial^2 v}{\partial x^2} &= e^{A\tau+Cx}\left(C^2u+2C\,\frac{\partial u}{\partial x}+\frac{\partial^2 u}{\partial x^2}\right),\end{aligned}$$

and substituting this into (10) and cancelling the common term $e^{A\tau+Cx}$ gives

$$\frac{\partial u}{\partial x}+A\,u=\tfrac{1}{2}\,\frac{\partial^2 u}{\partial x^2}+(C+k_1)\,\frac{\partial u}{\partial x}+\left(\tfrac{1}{2}C^2+k_1\,C-k_2\right)u.$$

By choosing

$$C=-k_1,\quad A=\tfrac{1}{2}C^2+k_1\,C-k_2=-\tfrac{1}{2}\,k_1^2-k_2$$

this reduces to the heat equation

$$\frac{\partial u}{\partial \tau}=\tfrac{1}{2}\,\frac{\partial^2 u}{\partial x^2}. \tag{13}$$

The initial condition (11) becomes

$$e^{Cx}\,u(x,0)=p(x)$$

and so

$$u(x,0)=e^{-Cx}\,p(x). \tag{14}$$

Suppose that $u(x,\tau)$ is the solution to (12) and set $\hat{u}(x,\tau)=u(2b-x,\tau)$ for some constant b. Show that $\hat{u}(x,\tau)$ is also a solution of the heat equation (but not necessarily the initial condition) in (12).

Assume that $u(x,\tau)$ satisfies (13) and set $\hat{u}(x,\tau)=u(\hat{x},\tau)$ where $\hat{x}=2b-x$ and b is a constant. It follows from (13) that

$$\frac{\partial u}{\partial \tau}(\hat{x},\tau)=\tfrac{1}{2}\,\frac{\partial^2 u}{\partial \hat{x}^2}(\hat{x},\tau).$$

Clearly

$$\frac{\partial}{\partial x}=\frac{\partial \hat{x}}{\partial x}\,\frac{\partial}{\partial \hat{x}}=-\frac{\partial}{\partial \hat{x}},\quad \frac{\partial^2}{\partial x^2}=\frac{\partial^2}{\partial \hat{x}^2}$$

and so

$$\frac{\partial \hat{u}}{\partial \tau}(x,\tau)=\frac{\partial u}{\partial \tau}(\hat{x},\tau),\quad \frac{\partial^2 \hat{u}}{\partial x^2}(x,\tau)=\frac{\partial^2 u}{\partial \hat{x}^2}(\hat{x},\tau).$$

Therefore

$$\frac{\partial u}{\partial \tau}(\hat{x},\tau) = \tfrac{1}{2}\,\frac{\partial^2 u}{\partial \hat{x}^2}(\hat{x},\tau) \quad \Longleftrightarrow \quad \frac{\partial \hat{u}}{\partial \tau}(x,\tau) = \tfrac{1}{2}\,\frac{\partial^2 \hat{u}}{\partial x^2}(x,\tau),$$

that is $\hat{u}(x,\tau) = u(2b - x, \tau)$ also satisfies the heat equation.

Unwinding the transformations that reduced the Black-Scholes equation to the heat equation it is clear that $u(x,\tau)$ leads back to the solution of the original Black-Scholes problem. Show that unwinding the transformations on $\hat{u}(x,\tau)$ leads to the 'reflected' solution

$$\hat{V}(S,t) = \left(\frac{S}{B}\right)^{2\alpha} V\Big(\frac{B^2}{S}, t\Big),$$

where $2\alpha = 1 - 2(r-y)/\sigma^2$ and $B > 0$.

Set

$$\begin{aligned} V(S,t) &= S_0\, e^{A\tau + Cx}\, u(x,\tau), \\ \hat{V}(S,t) &= S_0\, e^{A\tau + Cx}\, u(2b - x,\tau), \end{aligned}$$

where

$$x = \log\big(S/S_0\big), \quad S = S_0\, e^x.$$

From the first and third of these expressions we have

$$u(x,\tau) = e^{-A\tau - Cx}\, V\big(\,S_0 e^x, t\big)/S_0$$

and hence

$$u(2b - x,\tau) \;=\; e^{-A\tau - 2Cb + Cx}\, V\big(S_0 e^{2b-x}, t\big)/S_0.$$

Therefore

$$\begin{aligned} \hat{V}(S,t) &= S_0\, e^{A\tau + Cx}\, u\big(2b - x,\tau\big) \\ &= S_0\, e^{A\tau + Cx}\, e^{-A\tau - 2Cb + Cx}\, V\big(S_0 e^{2b-x}, t\big)/S_0 \\ &= e^{2C(x-b)}\, V\big(S_0\, e^{2b-x}, t\big). \end{aligned}$$

If we now put $b = \log(B/S_0)$ for some $B > 0$ then

$$e^{2C(x-b)} = \exp\big(\,2C\log(S/B)\,\big) = \left(\frac{S}{B}\right)^{2C}$$

and as

$$2C = -2k_1 = 1 - 2(r-y)/\sigma^2 = \beta,$$

(as earlier, set $\beta = 2\alpha$) we can write this as

$$e^{2C(x-b)} = \left(\frac{S}{B}\right)^{\beta}.$$

We also have

$$S_0\, e^{2b-x} = S_0\, \exp\big(\, 2\log(B) - \log(S) - \log(S_0)\,\big) = \frac{B^2}{S}$$

and thus we have

$$\hat{V}(S,t) = \left(\frac{S}{B}\right)^{\beta} V\left(\frac{B^2}{S}, t\right),$$

where $\beta = 1 - 2(r-y)/\sigma^2$.

10. The covariation of two functions or processes, X and Y, on $[0,t]$ is defined to be

$$[X,Y]_t = \lim_{|\pi|\to 0} \sum_{k=0}^{n-1} (X_{k+1} - X_k)(Y_{k+1} - Y_k).$$

Show that if both X and Y have finite quadratic variation on $[0,t]$ then $[X,Y]_t$ is finite and satisfies $2\big|\,[X,Y]_t\,\big| \le [X]_t + [Y]_t$.

Since $(x+y)^2 \ge 0$ and $(x-y)^2 \ge 0$, it follows that $2\,|x\,y| \le x^2 + y^2$ for any real numbers x and y.

So, with $\delta X_k = (X_{k+1} - X_k)$ and $\delta Y_k = (Y_{k+1} - Y_k)$,

$$2\,\big|\,\delta X_k\,\delta Y_k\,\big| \le \big(\,\delta X_k\,\big)^2 + \big(\,\delta Y_k\,\big)^2$$

and hence

$$\begin{aligned} 2\left|\sum_{k=0}^{n-1} \delta X_k\,\delta Y_k\right| &\le 2\sum_{k=0}^{n-1}\big|\,\delta X_k\,\delta Y_k\,\big| \\ &\le \sum_{k=0}^{n-1}\big((\delta X_k)^2 + (\delta Y_k)^2\big) \\ &= \left(\sum_{k=0}^{n-1}(\delta X_k)^2\right) + \left(\sum_{k=0}^{n-1}(\delta Y_k)^2\right). \end{aligned}$$

In the limit $|\pi| \to 0$ this shows that

$$2\,\big|\,[X,Y]_t\,\big| \le [X]_t + [Y]_t.$$

Assuming $[X]_t$ and $[Y]_t$ are finite, show that

(a) $[X+Y]_t = [X]_t + [Y]_t + 2\,[X,Y]_t$,

$$\sum_{k=0}^{n-1}\big(\,\delta X_k+\delta Y_k\,\big)^2$$

$$= \sum_{k=0}^{n-1}\Big(\,\big(\delta X_k\big)^2+\big(\delta Y_k\big)^2+2\,\delta X_k\,\delta Y_k\,\Big)$$

$$= \Big(\sum_{k=0}^{n-1}\big(\delta X_k\big)^2\Big)+\Big(\sum_{k=0}^{n-1}\big(\delta Y_k\big)^2\Big)+2\Big(\sum_{k=0}^{n-1}\,\delta X_k\,\delta Y_k\Big)$$

and taking the limit $|\pi|\to 0$ gives the result.

(b) $[X,Y]_t=\frac{1}{4}\big([X+Y]_t-[X-Y]_t\big)$,
From their definitions $[-Y]_t=[Y]_t$ and $[X,-Y]_t=-[X,Y]_t$, so

$$\begin{aligned}[X+Y]_t &= [X]_t+[Y]_t+2\,[X,Y],\\ [X-Y]_t &= [X]_t+[Y]_t-2\,[X,Y].\end{aligned}$$

Subtracting the second from the first then dividing by 4 gives the result. As a bonus, adding the two shows that

$$[X+Y]_t+[X-Y]_t=2\,[X]_t+2\,[Y]_t.$$

(c) if X and Y are C^1 functions on $[0,t]$ then $[X,Y]_t=0$.
If X and Y are C^1 then so too are $X+Y$ and $X-Y$. But we know that the quadratic variation of a C^1 function is zero, hence $[X+Y]_t=0$ and $[X-Y]_t=0$ and so the result follows from (b).

11. Let $(W_t)_{t\geq 0}$ and $(Z_t)_{t\geq 0}$ be two Brownian motions. They are correlated with correlation $\rho\in(-1,1)$ if

(a) for all $s,t\geq 0$, $\mathbb{E}\big[(W_{t+s}-W_t)(Z_{t+s}-Z_t)\big]=\rho\,s$,

(b) for all $0\leq p\leq q\leq s\leq t$, the pair (W_q-W_p) and (Z_t-Z_s) are independent and the pair (W_t-W_s) and (Z_q-Z_p) are also independent.

Show that in this case $[W,Z]_t=\rho\,t$, in the sense that

$$\mathbb{E}\big[[W,Z]_t-\rho\,t\big]=0\quad\text{and}\quad\mathbb{E}\Big[\big([W,Z]_t-\rho\,t\big)^2\Big]=0.$$

[Hint: first show that if X and Y are random variables with second moments then $|\,\mathbb{E}[\,XY\,]\,|\leq\frac{1}{2}\big(\mathbb{E}[X^2]+\mathbb{E}[Y^2]\big)$.]

As in the previous question, $|\,xy\,|\leq\frac{1}{2}x^2+\frac{1}{2}y^2$ for any real variables x and y. Therefore for any real random variables X and Y with second moments

$$\big|\,\mathbb{E}[\,XY\,]\,\big|\leq\mathbb{E}\big[\,|XY|\,\big]\leq\tfrac{1}{2}\mathbb{E}\big[\,X^2+Y^2\,\big]=\tfrac{1}{2}\mathbb{E}\big[\,X^2\,\big]+\tfrac{1}{2}\mathbb{E}\big[\,Y^2\,\big].$$

First consider

$$\begin{aligned}
\mathbb{E}\Big[\Big(\sum_{k=0}^{n-1}\delta W_k\,\delta Z_k\Big)-\rho\,t\Big] &= \mathbb{E}\big[\sum_{k=0}^{n-1}\big(\delta W_k\,\delta Z_k-\rho\,\delta t_k\big)\big]\\
&= \sum_{k=0}^{n-1}\mathbb{E}\big[\,\delta W_k\,\delta Z_k-\rho\,\delta t_k\,\big]\\
&= \sum_{k=0}^{n-1}\big(\,\mathbb{E}\big[\,\delta W_k\,\delta Z_k\,\big]-\rho\,\delta t_k\,\big) \;=\; 0.
\end{aligned}$$

This is true for any nontrivial partition π and so it is true in the limit $|\pi|\to 0$. Next consider

$$\begin{aligned}
\mathbb{E}\Big[\sum_{k=0}^{n-1}\Big(\big(\delta W_k\,\delta Z_k-\rho\,\delta t_k\big)\Big)^2\Big] &= \mathbb{E}\Big[\sum_{j,k=0}^{n-1}\big(\delta W_j\,\delta Z_j-\rho\,\delta t_j\big)\big(\delta W_k\,\delta Z_k-\rho\,\delta t_k\big)\Big]\\
&= \mathbb{E}\Big[\sum_{k=0}^{n-1}\big(\delta W_k\,\delta Z_k-\rho\,\delta t_k\big)^2\Big],
\end{aligned}$$

using the independence of δW_j, δW_k, δZ_j and δZ_k for $j\neq k$. Thus

$$\begin{aligned}
&\mathbb{E}\Big[\sum_{k=0}^{n-1}\Big(\big(\delta W_k\,\delta Z_k-\rho\,\delta t_k\big)\Big)^2\Big]\\
&\quad= \sum_{k=0}^{n-1}\mathbb{E}\Big[\big(\delta W_k\big)^2\big(\delta Z_k\big)^2-2\,\rho\,\delta t_k\,\delta W_k\,\delta Z_k+\big(\rho\,\delta t_k\big)^2\Big]\\
&\quad= \sum_{k=0}^{n-1}\Big(\mathbb{E}\big[\big(\delta W_k\big)^2\big(\delta Z_k\big)^2\big]-2\,\rho\,\delta t_k\,\mathbb{E}\big[\delta W_k\,\delta Z_k\big]+\big(\rho\,\delta t_k\big)^2\Big)\\
&\quad= \sum_{k=0}^{n-1}\Big(\mathbb{E}\big[\big(\delta W_k\big)^2\big(\delta Z_k\big)^2\big]-\big(\rho\,\delta t_k\big)^2\Big).
\end{aligned}$$

Now observe that

$$\Big|\,\mathbb{E}\big[\big(\delta W_k\big)^2\big(\delta Z_k\big)^2\big]\,\Big| \le \tfrac{1}{2}\mathbb{E}\big[\big(\delta W_k\big)^4\big]+\tfrac{1}{2}\mathbb{E}\big[\big(\delta Z_k\big)^4\big]=3\,(\delta t_k)^2,$$

and so, since $\rho\in(-1,1)$ implies $3-\rho^2\geq 2$,

$$\begin{aligned}
0 \;\le\; \mathbb{E}\Big[\sum_{k=0}^{n-1}\Big(\big(\delta W_k\,\delta Z_k-\rho\,\delta t_k\big)\Big)^2\Big] &\le \sum_{k=0}^{n-1}\big(3-\rho^2\big)\,\big(\delta t_k\big)^2\\
&\le (3-\rho^2)\,|\pi|\sum_{k=0}^{n-1}\delta t_k\\
&= (3-\rho^2)\,|\pi|\,t.
\end{aligned}$$

Taking the limit $|\pi| \to 0$ gives the result.

[Note that if we define a process by $f_t = f(W_t, Z_t, t)$ where $f(W, Z, t)$ is $C^{2,2,1}$, then (the differential version of) Itô's lemma is

$$\begin{aligned} df_t &= \frac{\partial f}{\partial t}\, dt + \frac{\partial f}{\partial W}\, dW_t + \frac{\partial f}{\partial Z}\, dZ_t \\ &+ \tfrac{1}{2}\frac{\partial^2 f}{\partial W^2}\, d[W]_t + \tfrac{1}{2}\frac{\partial^2 f}{\partial Z^2}\, d[Z]_t + \frac{\partial^2 f}{\partial W \partial Z}\, d[W, Z]_t, \end{aligned}$$

where all functions on the right-hand side are evaluated at (W_t, Z_t, t). The result derived above simplifies this expression.]

This is simply a comment.