B8.3 Mathematical Models for Financial Derivatives

Hilary Term 2019

Problem Sheet Four Solutions

Your grade will be determined from the *best five* answers to the first *seven* questions.

1. The price of a share evolves according to

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW_t, \quad S_0 > 0,$$

which implies that the share does *not* pay any dividends. There is a constant, risk-free, continuously-compounded interest rate r. A non-standard European derivative security is written on this share; in addition to paying the up-front price of the derivative the holder of the claim must also pay an amount $\beta\, S_t^\alpha\, dt$ over each interval $[t, t + dt)$ during the life of the derivative.[1] Let the derivative security's price function be $V(S, t)$, for $S > 0$ and $t \leq T$.[2] By suitably adapting either the delta-hedging or the self-financing replication argument, show that $V(S, t)$ must satisfy the partial differential equation

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2\, S^2\, \frac{\partial^2 V}{\partial S^2} + r\, S\, \frac{\partial V}{\partial S} - r\, V = \beta\, S^\alpha.$$

Delta-hedging argument: Form a portfolio long one option and short Δ_t shares. The market value of the portfolio is $M_t = V_t - \Delta_t\, S_t$ and the change in the hedging cost is

$$d\Pi_t = dV_t - \Delta_t\, dS_t - \beta\, S_t^\alpha\, dt$$

the extra term (in red) being because the holder of the derivative security has to pay $\beta\, S_t^\alpha\, dt$ during the interval $[t, t + dt)$. Itô's lemma for $V_t = V(S_t, t)$ gives

$$dV_t = \left(\frac{\partial V}{\partial t}(S_t, t) + \tfrac{1}{2}\sigma^2\, S_t^2\, \frac{\partial^2 V}{\partial S^2}(S_t, t)\right) dt + \frac{\partial V}{\partial S}(S_t, t)\, dS_t.$$

Substituting into the expression for $d\Pi_t$ gives

$$d\Pi_t = \left(\frac{\partial V}{\partial t}(S_t, t) + \tfrac{1}{2}\sigma^2\, S_t^2\, \frac{\partial^2 V}{\partial S^2}(S_t, t) - \beta\, S^\alpha\right) dt + \left(\frac{\partial V}{\partial S}(S_t, t) - \Delta_t\right) dS_t$$

[1] Here α is a real number and β has dimensions of $[\text{price}]^{1-\alpha}/[\text{time}]$.

[2] This means that if S_t is the share's price then the derivative's price is $V_t = V(S_t, t)$.

and at time t the only random term here is dS_t. If we set

$$\Delta_t = \frac{\partial V}{\partial S}(S_t, t)$$

the change in the hedging cost becomes deterministic (i.e., not random). At this point we could either hold the portfolio until time $t + dt$ or we could sell it for the market price, put the money in the bank and earn interest. As both of these strategies are risk-free, we must have

$$d\Pi_t = r\, M_t\, dt$$

or there would be an arbitrage opportunity. Written out in full this becomes

$$\left(\frac{\partial V}{\partial t}(S_t, t) + \tfrac{1}{2}\sigma^2\, S_t^2\, \frac{\partial^2 V}{\partial S^2}(S_t, t) - \beta\, S_t^\alpha\right) dt = r\left(V(S_t, t) - S_t\, \frac{\partial V}{\partial S}(S_t, t)\right) dt.$$

Cancelling the the dt term and rearranging gives

$$\frac{\partial V}{\partial t}(S_t, t) + \tfrac{1}{2}\sigma^2\, S_t^2\, \frac{\partial^2 V}{\partial S^2}(S_t, t) + r\, S_t\, \frac{\partial V}{\partial S}(S_t, t) - r\, V(S_t, t) = \beta\, S_t^\alpha.$$

Now recall that viewed from time zero, say, we have

$$S_t = S_0\, e^{(r - \sigma^2/2)t + \sigma W_t}$$

so it follows that S_t could take any positive value (because W_t can take any real value). Thus we way as well drop the subscript on S_t and just write S, in which case we have the (extended) Black-Scholes equation

$$\frac{\partial V}{\partial t}(S, t) + \tfrac{1}{2}\sigma^2\, S^2\, \frac{\partial^2 V}{\partial S^2}(S, t) + r\, S\, \frac{\partial V}{\partial S}(S, t) - r\, V(S, t) = \beta\, S^\alpha,$$

which holds for all $S > 0$ and $t < T$.

Self-financing replication argument:

Let S_t denote the price of a share and B_t the price of a bond. We have

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW_t, \quad \frac{dB_t}{B_t} = r\, dt.$$

The solution of the bond-pricing equation is simply $B_t = B_0\, e^{rt}$ and so any function of the bond price can just as easily be written as a function of t (which is what we will do). Construct a portfolio consisting of $\phi_t = \phi(S_t, t)$ shares and $\psi_t = \psi(S_t, t)$ bonds. The value of the portfolio at time t is

$$\Phi_t = \psi_t\, S_t + \psi_t\, B_t,$$

which could also be written as

$$\Phi_t = \Phi(S_t, t) = \phi(S_t, t)\, S_t + \psi(S_t, t)\, B_0\, e^{rt}.$$

We find, from definition, that

$$\begin{aligned}
d\Phi_t &= \Phi_{t+dt} - \Phi_t \\
&= (\phi_t + d\phi_t)(S_t + dS_t) + (\psi_t + d\psi_t)(B_t + dB_t) - \phi_t\, S_t - \psi_t\, B_t \\
&= \phi_t\, dS_t + \psi_t\, dB_t + (S_t + dS_t)\, d\phi_t + (B_t + dB_t)\, d\psi_t.
\end{aligned}$$

The self-financing condition is now

$$(S_t + dS_t)\, d\phi_t + (B_t + dB_t)\, d\psi_t = +\beta\, S_t^\alpha\, dt,$$

the term in red being because we have to *pay* out $\beta\, S_t^\alpha\, dt$ if we hold the derivative security and we fund this by the difference between what we sell in shares and what we buy in bonds. Thus we have

$$\begin{aligned}
d\Phi_t &= \phi_t\, dS_t + \psi_t\, dB_t + \beta\, S_t^\alpha\, dt \\
&= \left(r\, \psi_t\, B_t + \beta\, S_t^\alpha\right) dt + \phi_t\, dS_t.
\end{aligned}$$

Next we note that we can also write $\Phi_t = \Phi(S_t, t)$ and use Itô's lemma to deduce that

$$d\Phi_t = \left(\frac{\partial \Phi}{\partial t}(S_t, t) + \tfrac{1}{2}\sigma^2\, S_t^2\, \frac{\partial^2 \Phi}{\partial S^2}(S_t, t)\right) dt + \frac{\partial \Phi}{\partial S}(S_t, t)\, dS_t.$$

Viewing these equations at time t, the only random term is dS_t and so we must have

$$\phi_t = \frac{\partial \Phi}{\partial S}(S_t, t)$$

and then, from the dt terms, we must also have

$$\frac{\partial \Phi}{\partial t}(S_t, t) + \tfrac{1}{2}\sigma^2\, S_t^2\, \frac{\partial^2 \Phi}{\partial S^2}(S_t, t) = r\, B_t\, \psi_t + \beta\, S_t^\alpha.$$

From the definition $\Phi_t = \phi_t\, S_t + \psi_t\, B_t$ we have

$$\begin{aligned}
\psi_t\, B_t &= \Phi_t - \phi_t\, S_t \\
&= \Phi(S_t, t) - S_t\, \frac{\partial \Phi}{\partial S}(S_t, t)
\end{aligned}$$

and so, rearranging slightly, we end up with

$$\frac{\partial \Phi}{\partial t}(S_t, t) + \tfrac{1}{2}\sigma^2\, S_t^2\, \frac{\partial^2 \Phi}{\partial S^2}(S_t, t) + r\, S_t\, \frac{\partial \Phi}{\partial S}(S_t, t) - r\, \Phi(S_t, t) = \beta\, S_t^\alpha.$$

View from, say, time zero this is true for all positive values of S_t and so we may as well just write it as

$$\frac{\partial \Phi}{\partial t}(S,t) + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 \Phi}{\partial S^2}(S,t) + r\,S\,\frac{\partial \Phi}{\partial S}(S,t) - r\,\Phi(S,t) = \beta\, S^\alpha$$

where $S > 0$. For $t < T$ this portfolio has exactly the same cash-flows as the derivative security, namely $\beta\, S_t^\alpha\, dt$. So, if we insist that it replicates the derivative security's payoff,

$$\Phi(S,T) = V(S,T)$$

then it perfectly replicates the derivative security's cash-flows for all time $t \le T$ and so (by no-arbitrage) it must have the same value as the derivative security, i.e, $V(S,t) = \Phi(S,t)$ for all $S > 0$ and $t \le T$. Thus

$$\frac{\partial V}{\partial t}(S,t) + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}(S,t) + r\,S\,\frac{\partial V}{\partial S}(S,t) - r\,V(S,t) = \beta\, S^\alpha.$$

(a) Assume first that $\beta = 0$. Find all separable solutions of the form $V(S,t) = f(t)\, S^m$ where m is a real constant. Assume that $f(T) = 1$.

We have

$$\frac{\partial V}{\partial t} = \dot f(t)\, S^m, \quad S\frac{\partial V}{\partial S} = m\, f(t)\, S^m, \quad S^2 \frac{\partial^2 V}{\partial S^2} = m(m-1)\, f(t)\, S^m.$$

Substituting into the Black-Scholes equation

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r\,S\,\frac{\partial V}{\partial S} - r\,V = 0$$

gives

$$\Big(\dot f(t) + \tfrac{1}{2}\sigma^2 m\,(m-1)\, f(t) + r\, m\, f(t) - r\, f(t)\Big)\, S^m = 0.$$

As this has to hold for all $S > 0$, it follows that

$$\dot f(t) + \lambda\, f(t) = 0,$$

where

$$\lambda = \left(\tfrac{1}{2}\sigma^2\, m + r\right)(m-1)\,.$$

Given that $f(T) = 1$, the solution must be $f(t) = e^{\lambda(T-t)}$ and hence

$$V(S,t) = S^m\, e^{\lambda(T-t)}, \quad \lambda = \left(\tfrac{1}{2}\sigma^2\, m + r\right)(m-1)\,.$$

(b) Find the steady-state solutions of this equation, that is, find solutions $V(S)$ that only depend on S. Assume here that $\alpha \neq 1$, $\alpha \neq -2r/\sigma^2$ and $\beta \neq 0$.

In this case $\partial V/\partial t = 0$ and the equation reduces to

$$\tfrac{1}{2}\sigma^2 S^2 \frac{d^2V}{dS^2} + r\,S\,\frac{dV}{dS} - r\,V = \beta\,S^\alpha.$$

The equation is linear so we can write the solution as

$$V(S) = V_c(S) + V_p(S)$$

where the complementary function, $V_c(S)$, satisfies

$$\tfrac{1}{2}\sigma^2 S^2 \frac{d^2V_c}{dS^2} + r\,S\,\frac{dV_c}{dS} - r\,V = 0.$$

and the particular integral, $V_p(S)$, is one solution of the original equation. One trick to find the complementary function is to assume that $V_c(S) = S^c$ for some constant c. This gives

$$S\,V_c'(S) = c\,S^c, \quad S^2\,V_c''(S) = c\,(c-1)\,S^c$$

and so we get

$$\left(\tfrac{1}{2}\sigma^2\,c(c-1) + r\,c - r\right) S^c = 0.$$

The only way this can be true for all $S > 0$ is if

$$\left(\tfrac{1}{2}\sigma^2\,c(c-1) + r\,c - r\right) = \left(\tfrac{1}{2}\sigma^2\,c + r\right)(c-1) = 0$$

which shows that either $c = 1$ or $c = -2r/\sigma^2$. As the equation is linear, the general solution is[3]

$$V_c(S) = A\,S + B\,S^{-2r/\sigma^2},$$

where A and B are constants.

In view of the above, to find a particular integral simply look for a solution of the form $V_p(S) = C\,S^\alpha$. We find that

$$S\,V_p'(S) = C\,\alpha\,S^\alpha, \quad S^2\,V_p''(S) = C\,\alpha(\alpha-1)\,S^\alpha$$

and hence

$$C\left(\tfrac{1}{2}\sigma^2\,\alpha(\alpha-1) + r\,\alpha - r\right) S^\alpha = \beta\,S^\alpha.$$

[3]Note that this shows that with no dividends S is a solution of the Black-Scholes equation. This is actually a reality check as a simple no-arbitrage argument shows that S must be solution of *any* equation which purports to price options if there are no dividends.

Cancelling the terms S^α this reduces to an equation for C, namely

$$C\left(\tfrac{1}{2}\sigma^2\,\alpha + r\right)(\alpha - 1) = \beta.$$

As we assume that $\alpha \neq 1$ and $\alpha \neq -2r/\sigma^2$, this can be solved for C to give

$$C = \frac{\beta}{\left(\tfrac{1}{2}\sigma^2\,\alpha + r\right)(\alpha - 1)}.$$

This gives the general solution

$$\begin{aligned} V(S) &= V_c(S) + V_p(S) \\ &= A\,S + B\,S^{-2r/\sigma^2} + C\,S^\alpha, \end{aligned}$$

where C is as given above.

(c) Without doing the details, briefly explain how you would solve the problem

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2\,S^2\,\frac{\partial^2 V}{\partial S^2} + r\,S\,\frac{\partial V}{\partial S} - r\,V = \beta\,S^\alpha,$$
$$V(S,T) = S^3,$$

assuming again that $\alpha \neq 1$, $\alpha \neq -2r/\sigma$ and $\beta \neq 0$.

The equation is linear so we can add solutions. Write $V(S,t) = V_1(S) + V_2(S,t)$ and use the steady-state solution, $V_1(S)$, to deal with the $\beta\,S^\alpha$ term on the right-hand side of the equation. We may as well just take $V_1(S)$ to be the particular solution given in the previous part of the question (i.e., take $A = 0$ and $B = 0$ in the previous part of the question) as this makes the calculations easier. Thus

$$V_1(S) = C\,S^\alpha.$$

Now observe that $V_2(S,t) = V(S,t) - V_1(S)$ and, in particular, that

$$\begin{aligned} V_2(S,T) &= V(S,T) - V_1(S) \\ &= S^3 - C\,S^\alpha. \end{aligned}$$

The solution to this problem (for $V_2(S,t)$) can be written as a sum of separable solutions using the results given in part (a) above.

2. Suppose that $V(S,t)$ satisfies the Black-Scholes problem

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-y)\, S\, \frac{\partial V}{\partial S} - r\, V = 0, \quad S > 0,\ t < T,$$
$$V(S,T) = P_{\mathrm{o}}(S), \quad S > 0.$$

Use the chain rule to show that if $F = S\, e^{(r-y)(T-t)}$ (the forward price of S over the time interval $[t,T]$), $t' = t$ and $\hat{V}(F,t') = V(S,t)$ then

$$\frac{\partial \hat{V}}{\partial t'} + \tfrac{1}{2}\sigma^2 F^2 \frac{\partial^2 \hat{V}}{\partial F^2} - r\, \hat{V} = 0, \quad F > 0,\ t' < T,$$
$$\hat{V}(F,T) = P_{\mathrm{o}}(F), \quad F > 0.$$

Put $F = S\, e^{(r-y)(T-t)}$, $t' = t$ and $\hat{V}(F,t') = V(S,t)$. We have

$$\frac{\partial V}{\partial S} = \frac{\partial F}{\partial S}\frac{\partial \hat{V}}{\partial F} = e^{(r-y)(T-t)}\frac{\partial \hat{V}}{\partial F}$$

and hence

$$S\,\frac{\partial V}{\partial S} = S\, e^{(r-y)(T-t)}\frac{\partial \hat{V}}{\partial F} = F\,\frac{\partial \hat{V}}{\partial F}. \tag{1}$$

Similarly, we find that

$$S^2\frac{\partial^2 V}{\partial S^2} = F^2\frac{\partial^2 \hat{V}}{\partial F^2}. \tag{2}$$

We also have

$$\frac{\partial V}{\partial t} = \frac{\partial t'}{\partial t}\frac{\partial \hat{V}}{\partial t'} + \frac{\partial F}{\partial t}\frac{\partial \hat{V}}{\partial F} = \frac{\partial \hat{V}}{\partial t'} - (r-y)\, F\,\frac{\partial \hat{V}}{\partial F} \tag{3}$$

Substituting (1)–(3) into the Black-Scholes equation gives

$$\frac{\partial \hat{V}}{\partial t'} - (r-y)\, F\,\frac{\partial \hat{V}}{\partial F} + \tfrac{1}{2}\sigma^2 F^2\frac{\partial^2 \hat{V}}{\partial F^2} + (r-y)\, F\,\frac{\partial \hat{V}}{\partial F} - r\,\hat{V} = 0,$$

which clearly simplifies to

$$\frac{\partial \hat{V}}{\partial t'} + \tfrac{1}{2}\sigma^2 F^2\frac{\partial^2 \hat{V}}{\partial F^2} - r\,\hat{V} = 0,$$

and holds for $F > 0$ and $t' < T$. When $t = T$ we have $t' = T$ and $F = S > 0$ and so the terminal condition becomes

$$\hat{V}(F,T) = P_{\mathrm{o}}(F), \quad F > 0.$$

3. Consider the following perpetual American option problem. The option's payoff is

$$P_{\mathrm{o}}(S) = \begin{cases} K - S/3 & \text{if } 0 < S \leq K, \\ 0 & \text{if } S > K. \end{cases}$$

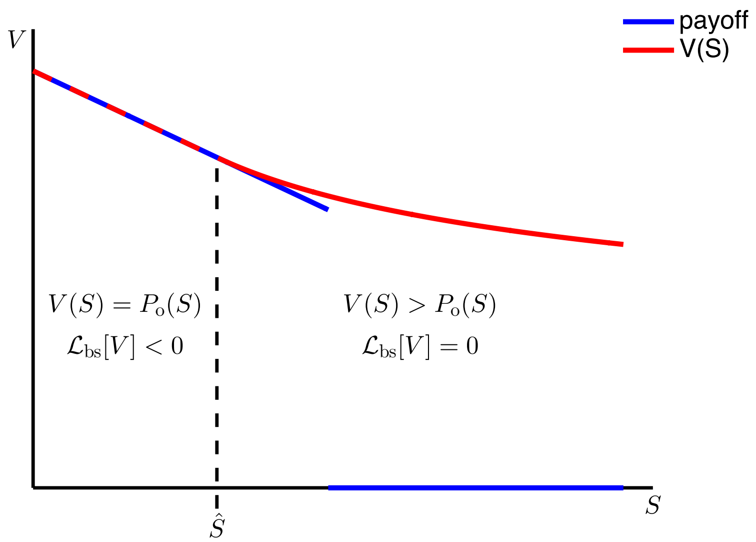
Assume that the option value satisfies the steady-state Black-Scholes equation

$$\mathcal{L}_{\mathrm{bs}}[V] = \tfrac{1}{2}\sigma^2\, S^2\, V''(S) + (r - y)\, S\, V'(S) - r\, V = 0, \quad \hat{S} < S,$$

where $0 < \hat{S} \leq K$ is the optimal exercise boundary and where $\sigma > 0$, $r > 0$ and $y > 0$ are constants. The option satisfies the boundary conditions

$$V(\hat{S}) = K - \hat{S}/3, \qquad \lim_{S\to\infty} V(S) \to 0.$$

(a) Give a *sketch* of the payoff and option price as functions of S and indicate where $\mathcal{L}_{\mathrm{bs}}[V] = 0$, where $\mathcal{L}_{\mathrm{bs}}[V] < 0$, where $V(S) > P_{\mathrm{o}}(S)$ and where $V(S) = P_{\mathrm{o}}(S)$.



(b) Prove that under the assumptions given above the quadratic

$$p(m) = \tfrac{1}{2}\sigma^2\, m(m-1) + (r - y)\, m - r$$

has two distinct real roots and only one of these is strictly negative.

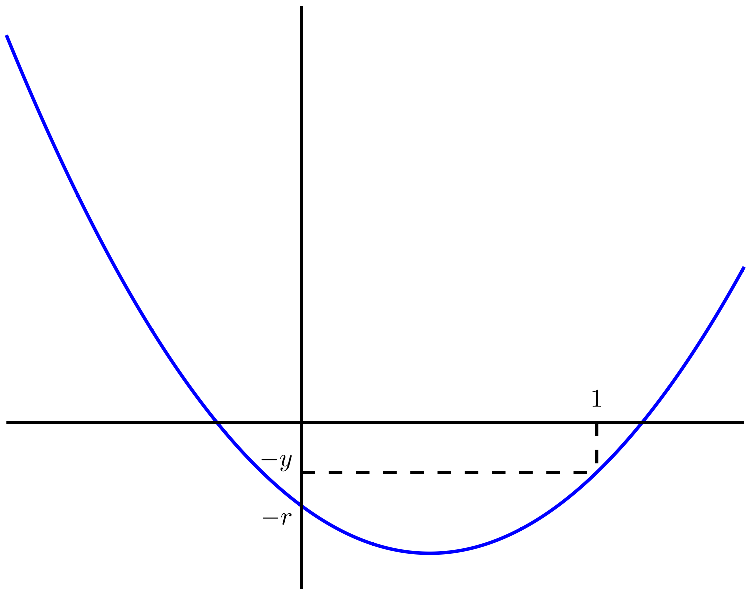
It is clear that

$$\lim_{m\to-\infty} p(m) \to \infty, \quad \lim_{m\to\infty} p(m) \to \infty$$

and that

$$p(0) = -r < 0, \quad p(1) = -y < 0.$$

Given that a quadratic can has only one turning point, these prove that the quadratic looks like



So one root is negative and the other is greater than one.

(c) Assume that we have smooth pasting at $\hat{S}$, i.e., $V'(\hat{S}) = -1/3$. Show that this implies that

$$\hat{S} = \frac{3m^-}{m^- - 1} K,$$

where $m^- < 0$ is the negative root of the quadratic $p(m)$.

First note that if we assume that $V(S) = S^m$ then m satisfies the quadratic equation

$$p(m) = \tfrac{1}{2}\sigma^2 m(m-1) + (r-y)\,m - r = 0$$

so there are two real roots, $m^- < 0$ and $m^+ > 1$. Thus the general solution of the ODE for $V(S)$ is

$$V(S) = A\,S^{m^-} + B\,S^{m^+}.$$

Second note that $V(\hat{S}) = K - \hat{S}/3$ and $\lim_{S\to\infty} V(S) \to 0$ imply that

$$V(S) = (K - \hat{S}/3)\left(\frac{S}{\hat{S}}\right)^{m^-}$$

for $S \geq \hat{S}$.

If the smooth pasting condition $V'(\hat{S}) = -1/3$ applies then we have

$$V'(S) = m^-\left(\frac{K - \hat{S}/3}{S}\right)\left(\frac{S}{\hat{S}}\right)^{m^-}$$

and so

$$V'(\hat{S}) = m^-\left(\frac{K - \hat{S}/3}{\hat{S}}\right) = -\frac{1}{3}.$$

When solved for $\hat{S}$ this gives

$$\hat{S} = \frac{3m^- K}{m^- - 1}.$$

(d) Show that smooth pasting only makes sense if $-\frac{1}{2} < m^- < 0$.

Clearly we need $\hat{S} > 0$ and $\hat{S} \leq K$. In the first case, the asset price can never reach $S = 0$ or $S < 0$ and so the option would never be exercised. In the latter case if $\hat{S} > K$ then we are exercising the option when its payoff is zero, i.e., for nothing, and this is clearly not optimal.

Thus we need

$$0 < \frac{3m^- K}{m^- - 1} \leq K$$

and since $K > 0$ this translates to

$$0 < \frac{3m^-}{m^- - 1} \leq 1$$

Given that we know $m^- < 0$ from Part (b) we automatically have

$$0 < \frac{3m^-}{m^- - 1}.$$

The other inequality, together with $m^- < 0$, gives

$$3m^- \geq m^- - 1,$$

which is equivalent to

$$m^- \geq -\tfrac{1}{2}.$$

(e) What is the optimal exercise boundary if $m^- < -\frac{1}{2}$? Justify your answer.

If $m^- < -\frac{1}{2}$ then we must have $\hat{S} = K$, i.e., the optimal exercise boundary is at the strike.

It is clear that we can't have $\hat{S} > K$ as this implies we exercise the option when the payoff is zero, which is clearly not optimal.

Suppose that we have $0 < \hat{S} < K$. Then, as above, we find that

$$V'(\hat{S}) = m^- \left(\frac{K - \hat{S}/3}{\hat{S}} \right) = m^- \left(\frac{K}{\hat{S}} - \tfrac{1}{3} \right) \leq \tfrac{2}{3}\, m^- < -\tfrac{1}{3}.$$

(Note that as $K/\hat{S} > 1$, it follows that $K/\hat{S} - \frac{1}{3} > \frac{2}{3}$ and hence that $m^-\big(K/\hat{S} - \frac{1}{2}\big) \leq \frac{2}{3}\, m^-$, because $m^- < 0$.) This means that the option's value falls below the payoff for S greater than but close to $\hat{S}$, which is an arbitrage for an American option.

Since both $\hat{S} < K$ and $\hat{S} > K$ are both impossible, the only option is $\hat{S} = K$. (When $\hat{S} = K$ we have, for $S > \hat{S}$, a non-zero value for the option which is always above the (zero) payoff.)

(f) Suppose that $-\frac{1}{2} < m^- < 0$, so that smooth pasting does give the correct optimal exercise boundary. Suppose also that the holder of the option decides that they are going to ignore the optimal exercise boundary $\hat{S}$ and simply exercise the option as soon as $S \leq \bar{S}$ where $0 < \bar{S} < K$ is chosen by the holder. In this case the value of the option, $\bar{V}(S, t)$, satisfies the problem

$$\mathcal{L}_{\mathrm{bs}}[\,\bar{V}\,] = 0, \quad S > \bar{S},$$
$$\bar{V}(\bar{S}) = K - \bar{S}/3, \quad \lim_{S\to\infty} \bar{V}(S) \to 0.$$

Find $\bar{V}(S)$ and show that

i. if $\bar{S} > \hat{S}$ then one could increase the value of the option by decreasing $\bar{S}$ (hint; differentiate with respect to $\bar{S}$);

ii. if $\bar{S} < \hat{S}$ then there is a potential arbitrage in the price $\bar{V}(S)$ (hint; differentiate with respect to S).

As above the general solution of the ODE is

$$\bar{V}(S; \bar{S}) = A\, S^{m^-} + B\, S^{m^+},$$

where $m^- < 0$, $m^+ \geq 1$. The condition that $\lim_{S\to\infty} \bar{V}(S) = 0$ shows that $B = 0$, so

$$\bar{V}(S; \bar{S}) = A\, S^{m^-}.$$

The condition $\bar{V}(\bar{S}; \bar{S}) = K - \bar{S}/3$ gives

$$A\, \bar{S}^{m^-} = K - \bar{S}/3 \quad \Longrightarrow \quad A = (K - \bar{S}/3)/\bar{S}^{m^-}$$

and hence

$$\bar{V}(S;\bar{S}) = (K - \bar{S}/3)\left(\frac{S}{\bar{S}}\right)^{m^-}.$$

It follows that

$$\begin{aligned}
\frac{\partial \bar{V}}{\partial S}(S;\bar{S}) &= m^- \frac{(K-\bar{S}/3)}{S}\left(\frac{S}{\bar{S}}\right)^{m^-} \\
\frac{\partial \bar{V}}{\partial \bar{S}}(S;\bar{S}), &= -\left(\frac{1}{3} + m^-\left(\frac{K}{\bar{S}} - \frac{1}{3}\right)\right)\left(\frac{S}{\bar{S}}\right)^{m^-}.
\end{aligned}$$

On the one hand, if $\bar{S} > \hat{S}$ then it follows that $K/\bar{S} < K/\hat{S}$. Noting that $m^- < 0$ and that $\hat{S}$ satisfies the equation

$$\frac{1}{3} + m^-\left(\frac{K - \hat{S}/3}{\hat{S}}\right) = \frac{1}{3} + m^-\left(\frac{K}{\hat{S}} - \frac{1}{3}\right) = 0,$$

it follows that

$$\frac{\partial \bar{V}}{\partial \bar{S}}(S;\bar{S}) < 0.$$

This implies that by *decreasing* the value of $\bar{S}$ we can *increase* the value of $\bar{V}$.

On the other hand, if $\hat{S} > \bar{S}$ then $K/\hat{S} < K/\bar{S}$. Then we see that

$$\begin{aligned}
\frac{\partial \bar{V}}{\partial S}(S;\bar{S}) &= m^-\left(\frac{K}{\bar{S}} - \frac{1}{3}\right) \\
&< m^-\left(\frac{K}{\hat{S}} - \frac{1}{3}\right) \qquad \text{(recall } m^- < 0\text{)} \\
&= -\frac{1}{3}.
\end{aligned}$$

This implies that the option's price falls *strictly below* the payoff (to the right of $\bar{S}$, near to $\bar{S}$), which is an arbitrage.

4. Let T_1 and T_2 be given times with $0 < T_1 < T_2$ and let $\alpha > 0$ be a given constant. A forward-start put is a European put option written on an asset whose price is S_t, but where the strike is not given at time zero, rather it is set equal to $\alpha\, S_{T_1}$, where S_{T_1} is the share price at time T_1. Find the option price for $T_1 < t < T_2$ and then for $0 \le t \le T_1$.

   For time $T_1 < t < T_2$ we know $K = \alpha\, S_{T_1}$ and so we have a regular put option. Its Black-Scholes value function is

$$P(S,t) = K\, e^{-r(T_2-t)}\,\mathrm{N}(-d_-) - S\, e^{-y(T_2-t)}\,\mathrm{N}(-d_+)$$

where

$$d_{\pm} = \frac{\log(S/K) + (r - y \pm \frac{1}{2}\sigma^2)(T_2 - t)}{\sqrt{\sigma^2(T_2 - t)}}, \quad K = \alpha\, S_{T_1}.$$

At time $t = T_1$ we have $S = S_{T_1}$ and $K = \alpha\, S = \alpha\, S_{T_1}$ by definition. Thus we have

$$P(S, T_1) = \alpha\, S\, e^{-r(T_2 - T_1)}\, \mathrm{N}(-\hat{d}_-) - S\, e^{-y(T_2 - T_1)}\, \mathrm{N}(-\hat{d}_+)$$

where

$$\hat{d}_{\pm} = \frac{\log(\alpha) + (r - y \pm \frac{1}{2}\sigma^2)(T_2 - T_1)}{\sqrt{\sigma^2(T_2 - T_1)}}.$$

Thus we can write

$$P(S, T_1) = S\, A(T_2, T_1, r, y, \sigma, \alpha)$$

where

$$A(T_2, T_1, r, y, \sigma, \alpha) = \alpha\, e^{-r(T_2 - T_1)}\, \mathrm{N}(-\hat{d}_-) - e^{-y(T_2 - T_1)}\, \mathrm{N}(-\hat{d}_+)$$

is independent of both S and t.

Solving that Black-Scholes equation backwards from T_1 we see that for $t \le T_1$ we have

$$P(S, t) = S\, e^{-y(T_1 - t)}\, A(T_2, T_1, r, y, \sigma, \alpha).$$

5. An up-and-out barrier put option is an option which has the payoff of a regular put option provided the share price stays below a barrier, $B > 0$, for the life of the option, i.e., provided $S_t < B$ for all $t \in [0, T]$. If at any time $t \in [0, T]$ we have $S_t \ge B$ the option immediately becomes worthless.

   (a) Write down the Black-Scholes problem for the price function of this option assuming that the share price has stayed below the barrier.

$$\frac{\partial P_{\mathrm{uo}}}{\partial t} + \tfrac{1}{2}\sigma^2\, S^2\, \frac{\partial^2 P_{\mathrm{uo}}}{\partial S^2} + (r - y)\, S\, \frac{\partial P_{\mathrm{uo}}}{\partial S} - r\, P_{\mathrm{uo}} = 0,\ 0 < S < B,\ t < T,$$

$$P_{\mathrm{uo}}(B, t) = 0, \quad t < T,$$

$$P_{\mathrm{uo}}(S, T) = (K - S)^+, \quad 0 < S < B.$$

(b) Find the Black-Scholes value function for this option in terms of a vanilla put's value function assuming that the barrier lies above the strike, $0 < K < B$.

Let

$$P(S,t) = K\,e^{-r(T-t)}\,\mathrm{N}(-d_-) - S\,e^{-y(T-t)}\,\mathrm{N}(-d_+),$$

where

$$d_\pm = \frac{\log(S/K) + (r - y \pm \frac{1}{2}\sigma^2)(T-t)}{\sqrt{\sigma^2(T-t)}},$$

be the Black-Scholes price function for a European put option. Using the reflection result, define the reflected value about the barrier B as

$$W(S,t) = \left(\frac{S}{B}\right)^\beta P\left(\frac{B^2}{S}, t\right), \quad \beta = 1 - 2(r-y)/\sigma^2.$$

We know that both $P(S,t)$ and $W(S,t)$ both satisfy the (same) Black-Scholes equation and therefore so too does their difference

$$P_{\mathrm{uo}}(S,t) = P(S,t) - W(S,t).$$

We also see that for any $t < T$ we have

$$\begin{aligned} P_{\mathrm{uo}}(B,t) &= P(B,t) - W(B,t) \\ &= P(B,t) - \left(\frac{B}{B}\right)^\beta P\left(\frac{B^2}{B}, t\right) \\ &= P(B,t) - P(B,t) \;=\; 0. \end{aligned}$$

Next note that for $0 < S < B$ we have

$$P_{\mathrm{uo}}(S,T) = P(S,T) - W(S,T) = (K-S)^+ - W(S,T).$$

Therefore, if we can show that $W(S,T) = 0$ for $0 < S < B$ then we have

$$P_{\mathrm{uo}}(S,T) = (K-S)^+$$

and thus $P_{\mathrm{uo}}(S,t)$ satisfies the problem for the up-and-out put option.

Now if $0 < S < B$ then $1 < B/S$ and hence $B < B^2/S$. We are also given that $K < B$ and therefore $K < B < B^2/S$, which implies that if $0 < S < B$ then

$$W(S,T) = \left(\frac{S}{B}\right)^\beta \left(K - \frac{B^2}{S}\right)^+ = 0,$$

since $K - B^2/S < 0$.

(c) Find the Black-Scholes value function for this option in terms of the price functions for vanilla and digital puts assuming the barrier lies below the strike, $0 < B < K$.

In the case $0 < B < K$ we have to truncate the put's payoff and reflect the function with payoff

$$V(S,T) = \begin{cases} K - S & \text{if } 0 < S < B, \\ 0 & \text{if } S \geq B, \end{cases}$$

about $S = B$. With the truncated payoff we have

$$P_{\mathrm{uo}}(S,t) = V(S,t) - \left(\frac{S}{B}\right)^{\beta} V\left(\frac{B^2}{S}, t\right), \quad 0 < S < B.$$

As above, we have $P_{\mathrm{uo}}(S,t)$ satisfies the Black-Scholes equation by linearity at $S = B$ we have

$$P_{\mathrm{uo}}(B,t) = V(B,t) - \left(\frac{B}{B}\right)^{\beta} V\left(\frac{B^2}{B}, t\right) = V(B,t) - V(B,t) = 0.$$

If we choose $0 < S < B$ then we have $B/S > 1$ and hence $B^2/S > B$, which means that

$$\left(\frac{S}{B}\right)^{\beta} V\left(\frac{B^2}{S}, T\right) = 0$$

as the $V(\cdot, T)$ term vanishes. Thus, for $0 < S < B$ we have

$$P_{\mathrm{uo}}(S,T) = V(S,T) = K - S,$$

as required. Finally note that $V(S,T)$ is simply a put with strike B plus $(K - B)$ times a digital put with strike B, i.e.,

$$V(S,T) = P(S,T; K = B) + (K - B)\, P_{\mathrm{d}}(S,T; K = B)$$

and hence

$$V(S,t) = P(S,t; K = B) + (K - B)\, P_{\mathrm{d}}(S,t; K = B).$$

(d) By analogy with the down-and-in call option, define an up-and-in put and find a formula for its value in the case $0 < K < B$.

The up-and-in put comes to life if the barrier is B is crossed, at which point it turns into a regular put option with strike K and expiry time T. That is, if there is some $t \leq T$ with $S_t > B$ then the up-and-in put becomes are regular put option with value $P(S,t; K, T)$. If there is not such t then the up-and-in barrier put expires worthless.

Since 'the barrier being crossed' and 'the barrier not being crossed' are mutually exclusive events and one or the other must happen, if we hold an up-and-out and an up-and-in barrier put (with the same barrier, strike and expiry date) then we must end up with a regular put option by expiry,

$$P_{\text{uo}}(S,T;K,T,B) + P_{\text{ui}}(S,T;K,T,B) = P(S,T;K,T).$$

By no arbitrage, this extends to

$$P_{\text{uo}}(S,t;K,T,B) + P_{\text{ui}}(S,t;K,T,B) = P(S,t;K,T).$$

Thus, if the up-and-in option has yet to come in (so the up-and-out option has yet to go out) then we have

$$P_{\text{ui}}(S,t;K,T,B) = P(S,t;K,T) - P_{\text{uo}}(S,t;K,T,B)$$

and if $B > K > 0$ then we can write this as

$$P_{\text{ui}}(S,t;K,T,B) = \left(\frac{S}{B}\right)^{\beta} P\left(\frac{B^2}{S},t\right), \quad 0 < S < B.$$

6. A down-and-out digital call option is a digital call option which becomes worthless if $S_t \leq B$ at any time during the options life, $[0,T]$. Here $B > 0$ is called the barrier. If we have $S_t > B$ for all $t \in [0,T]$ then the payoff for the option is the unit step function $\mathbf{1}_{\{S_T \geq K\}}$. If the underlying share pays a constant, continuous dividend yield y, find the Black-Scholes value of such an option if:

   (a) the barrier is less than the strike, $0 < B < K$;

   In this case we can write the solution in terms of the reflection of the digital call about $S = B$ and it is

$$C_{\text{ddo}}(S,t;K,T,B) = C_{\text{d}}(S,t;K,T) - \left(\frac{S}{B}\right)^{\beta} C_{\text{d}}\left(\frac{B^2}{S},t;K,T\right)$$

   for $B < S$, where $\beta = 1-2(r-y)/\sigma^2$. As in the previous question, it clearly satisfies the Black-Scholes equation by linearity and $C_{\text{ddo}}(B,t;K,T,B) = 0$. We also have $0 < B < K$ and so if $0 < B < S$ then $0 < B/S < 1$ and hence $0 < B^2/S < B < K$, which means that

$$C_{\text{d}}\left(\frac{B^2}{S},T;K,T\right) = \left(\frac{B^2}{S} - K\right)^{+} = 0.$$

   Therefore

$$C_{\text{ddo}}(S,T;K,T,B) = C_{\text{d}}(S,T;K,T) = (S-K)^{+}$$

   and so $C_{\text{ddo}}(S,t;K,T,B)$ satisfies all conditions in the Black-Scholes barrier problem.

(b) the barrier is greater than the strike, $0 < K < B$.

In this case we have to truncate the payoff so it becomes

$$V(S,T) = \begin{cases} 0 & \text{if } 0 < S \le B, \\ 1 & \text{if } S > B, \end{cases}$$

before we reflect about $S = B$. This is the same thing as changing the digital call's strike from K to B. That is, the solution in this case is

$$C_{\mathrm{ddo}}(S,t;B,T,B) = C_{\mathrm{d}}(S,t;B,T) - \left(\frac{S}{B}\right)^{\beta} C_{\mathrm{d}}\left(\frac{B^2}{S},t;B,T\right).$$

As above, the solution satisfies the Black-Scholes equation by linearity, it has the property that

$$C_{\mathrm{ddo}}(B,t;B,T,B) = 0.$$

For $0 < B < S$ we have $0 < B/S < 1$ and hence $0 < B^2/S < B$ which means that

$$\left(\frac{S}{B}\right)^{\beta} C_{\mathrm{d}}\left(\frac{B^2}{S},T\right) = 0$$

and hence that for $S > B > 0$ we have

$$C_{\mathrm{ddo}}(S,T;B,T,B) = C_{\mathrm{d}}(S,T;B,T) = 1,$$

which is the correct payoff.

7. If an option depends on a continuously sampled arithmetic average of the share price we can write its price as $V_t = V(S_t, R_t, t)$ where $R_t = \int_0^t S_u\,du$ and at expiry the average share price is $A_T = R_T/T$. The Black-Scholes equation for the function $V(S,R,T)$ is

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-y)\,S\,\frac{\partial V}{\partial S} + S\,\frac{\partial V}{\partial R} - r\,V = 0.$$

For $t < T$, find the solution of this equation if the payoff is

$$V(S,R,T) = A\,S + B\,R + C$$

for constants A, B and C. Assume that $r \neq y$.

[Hint: try a solution of the form $V(S,R,t) = a(t)\,S + b(t)\,R + c(t)$.]

As suggested, try a linear solution

$$V(S,R,t) = a(t)\,S + b(t)\,R + c(t).$$

At expiry this gives the terminal values

$$a(T) = A, \quad b(T) = B, \quad c(T) = C.$$

We have

$$\frac{\partial V}{\partial t} = \dot{a}(t)\, S + \dot{b}(t)\, R + \dot{c}(t), \quad \frac{\partial V}{\partial S} = a(t), \quad \frac{\partial V}{\partial R} = b(t), \quad \frac{\partial^2 V}{\partial S^2} = 0.$$

Substituting into the partial differential equation above and collecting terms gives

$$\big(\dot{a} - r\,a + (r-y)\,a + b\big)\, S + \big(\dot{b} - r\,b\big)\, R + \big(\dot{c} - r\,c\big) = 0$$

which simplifies to

$$\big(\dot{a} - y\,a + b\big)\, S + \big(\dot{b} - r\,b\big)\, R + \big(\dot{c} - r\,c\big) = 0.$$

Given that this holds for all $S > 0$ and $R > 0$ we must have the weakly coupled terminal value problem

$$\begin{aligned} \dot{a} - y\,a &= -b, & a(T) &= A, \\ \dot{b} &= r\,b, & b(T) &= B, \\ \dot{c} &= r\,c, & c(T) &= C. \end{aligned}$$

Clearly the final two equations give

$$b(t) = B\, e^{-r(T-t)}, \quad c(t) = C\, e^{-r(T-t)},$$

while the first can be written as

$$\frac{d}{dt}\big(\, e^{y(T-t)}\, a(t)\,\big) = -B\, e^{(y-r)(T-t)}, \quad a(T) = A.$$

This then integrates to give

$$A - e^{y(T-t)} a(t) = -B \int_t^T e^{(y-r)(T-u)}\, du$$

which (assuming $r \neq y$) gives

$$a(t) = A\, e^{-y(T-t)} - \frac{B}{r-y}\Big(e^{-r(T-t)} - e^{-y(T-t)} \Big)$$

Assume now that the payoff is $R_T = T \times A_T$. Explain how you could perfectly hedge such a contract. Is your method independent of the Black-Scholes equation?

The solution comes from putting $A = 0$, $B = 1$ and $C = 0$ in the previous part of the question, in which case

$$a(t) = \frac{e^{-y(T-t)} - e^{-r(T-t)}}{r-y}, \quad b(t) = e^{-r(T-t)}$$

and so

$$V_t = V(S_t, R_t, t) = \frac{\left(e^{-y(T-t)} - e^{-r(T-t)}\right) S_t}{r-y} + e^{-r(T-t)}\, R_t.$$

If this derivative security is perfectly hedged then we hold

$$\Delta_t = \frac{\partial V}{\partial S} = a(t) = \frac{e^{-y(T-t)} - e^{-r(T-t)}}{r-y}$$

shares. At expiry we have to pay out

$$R_T = \int_0^T S_u\, du = \int_0^t S_u\, du + \int_t^T S_u\, du = R_t + \int_t^T S_u\, du.$$

Clearly the term $e^{-r(T-t)}\, R_t$ in the derivative security's price will grow to R_t at time $t = T$ (assuming it is kept in the bank, which it is) and so we have only to worry about how we produce the term $\int_t^T S_u\, du$.

Over the time interval $[u, u + du) \subset [t, T)$ we sell exactly $e^{-r(T-u)} du$ shares and put the money in the bank, so $e^{-r(T-u)} S_u\, du$ is banked. At time $t = T$ this will be worth $S_u\, du$ and so, summing over all $u \in [t, T]$, our total amount banked is

$$\int_t^T S_u\, du$$

at expiry, as we need.

During the interval $[u, u + du)$ we also receive $y\, \Delta_u\, S_u\, du$ in dividends, allowing us to buy $y\, \Delta_u\, du$ new shares. Thus, over this interval we have the change in Δ_u given by

$$d\Delta_u = y\, \Delta_u\, du - e^{-r(T-u)}\, du$$

which gives the ordinary differential equation

$$\frac{d\Delta_u}{du} = y\, \Delta_u - e^{-r(T-u)},$$

and which can be written in terms of an integrating factor as

$$\frac{d}{du}\left(e^{y(T-u)}\, \Delta_u\right) = -e^{(y-r)(T-u)}. \tag{4}$$

If we integrate (4) subject to the condition that we have no shares left at T, as we don't need any shares for the payoff, we get

$$\Delta_t = \frac{e^{-y(T-t)} - e^{-r(T-t)}}{r-y},$$

which is exactly how many shares we do hold (if perfectly hedged).

This is a model independent result, it only depends on a constant interest rate r and a constant continuous dividend yield y. It does not depend on the model for S_t. It says that if you want to reproduce $\int_0^T S_u \, du$ then you start out with enough shares so that if you sell $e^{-r(T-t)} \, dt$ of them during each interval $[t, t + dt)$ then you end up with exactly zero shares at expiry, T.

Optional questions

8. For $t \in [0,T]$ let $S_t > 0$ satisfy the SDE

$$\frac{dS_t}{S_t} = (\mu - y)\,dt + \sigma\,dW_t, \quad S_0 = S > 0,$$

where μ, y and $\sigma > 0$ are constants. Let M_t denote the maximum value of S_u over the interval $[0,t]$,

$$M_t = \max_{0 \le u \le t} S_u,$$

for all $t \in [0,T]$. Show that

(a) M_t is a nondecreasing function of $t \in [0,T]$;
This follows from the fact that it is the maximum value of S_u over the interval $[0,t]$—by definition it can not get smaller as t increases.

(b) M_t is continuous in t;
It M_t is not continuous then S_t would have to be discontinuous, but we know that S_t is continuous (since

$$S_t = S\,\exp\big(\,(\mu - y - \tfrac{1}{2}\sigma^2)\,t + \sigma\,W_t\,\big)$$

and both t and W_t are continuous.)

(c) M_t has finite variation (and hence zero quadratic variation).
Using the above first point above we have

$$\begin{aligned} \langle M \rangle_t &= \lim_{|\pi|\to 0} \sum_{k=0}^{n-1} |\,M_{k+1} - M_k\,| \\ &= \lim_{|\pi|\to 0} \sum_{k=0}^{n-1} (M_{k+1} - M_k) \\ &= M_t - M_0, \end{aligned}$$

which is finite because M_t is continuous on $[0,T]$.

Deduce that the covariation $\langle S, M \rangle_t$, defined as

$$\langle S, M \rangle_t = \lim_{|\pi|\to 0} \sum_{k=0}^{n-1} (S_{k+1} - S_k)\,(M_{k+1} - M_k),$$

is zero for all $t \in [0,T]$. [Hint: note that although S_t has nonzero quadratic variation, and therefore infinite variation, it is nevertheless continuous in t.]

We have

$$
\begin{aligned}
\left|\,\langle S, M\rangle_t\,\right| &= \left|\lim_{|\pi|\to 0}\sum_{k=0}^{n-1}(S_{k+1}-S_k)\,(M_{k+1}-M_k)\right| \\
&\le \lim_{|\pi|\to 0}\sum_{k=0}^{n-1}|\,S_{k+1}-S_k\,|\,(M_{k+1}-M_k) \\
&\le \lim_{|\pi|\to 0}\Big(\max_{\pi}|\,S_{k+1}-S_k\,|\sum_{k=0}^{n-1}(M_{k+1}-M_k)\Big) \\
&= \Big(\lim_{|\pi|\to 0}\max_{\pi}|\,S_{k+1}-S_k\,|\Big)\Big(\lim_{|\pi|\to 0}\sum_{k=0}^{n-1}(M_{k+1}-M_k)\Big) \\
&= \Big(\lim_{|\pi|\to 0}\max_{\pi}|\,S_{k+1}-S_k\,|\Big)\,\langle M\rangle_t
\end{aligned}
$$

which vanishes since S_t is continuous in t, so

$$\lim_{|\pi|\to 0}\max_{\pi}|\,S_{k+1}-S_k\,| = 0,$$

and $\langle M\rangle_t$ is finite.

9. **Time-dependent parameters**

Assume that the price of a share, S_t, evolves according to the SDE

$$\frac{dS_t}{S_t} = \big(\,\bar{\mu}(t)-\bar{y}(t)\,\big)\,dt + \bar{\sigma}(t)\,dW_t,$$

where $\bar{\mu}(t)$, $\bar{y}(t)$ and $\bar{\sigma}(t) > 0$ are known functions of time. Assume also that the risk-free rate is a known function of time, $\bar{r}(t)$.

(a) Derive the Black-Scholes problem, for $S > 0$,

$$
\begin{gathered}
\frac{\partial C}{\partial t} + \tfrac{1}{2}\bar{\sigma}(t)^2 S^2\frac{\partial^2 C}{\partial S^2} + \big(\bar{r}(t)-\bar{y}(t)\big)S\frac{\partial C}{\partial S} - \bar{r}(t)C = 0,\ \ t < T, \\
V(S,T) = (S-K)^+,
\end{gathered}
\tag{5}
$$

for the value (function) $C(S,t)$ of a European call option written on the share.

This is exactly the same as the derivation of the Black-Scholes equation with constant r, y and σ (except, in this case, we write $r(t)$, $y(t)$ and $\sigma(t)$). The payoff follows in exactly the same way as it does for the constant parameter case.

(b) Use the Feynman-Kăc theorem to show that

$$C(S,t) = \exp\Big(-\int_t^T \bar{r}(u)\,du\Big)\,\mathbb{E}_t\big[\,(S_T-K)^+\,|\,S_t = S\,\big],$$

where S_t evolves as

$$\frac{dS_t}{S_t} = \big(\bar{r}(t) - \bar{y}(t)\big)\,dt + \bar{\sigma}(t)\,dW_t. \tag{6}$$

Write

$$C(S,t) = \exp\Big(-\int_t^T \bar{r}(u)\,du\Big)\,U(S,t)$$

so that

$$\frac{\partial C}{\partial t} = r(t)\,C + \exp\Big(-\int_t^T \bar{r}(u)\,du\Big)\,\frac{\partial U}{\partial t}$$

while

$$\frac{\partial C}{\partial S} = \exp\Big(-\int_t^T \bar{r}(u)\,du\Big)\,\frac{\partial U}{\partial S}, \quad \frac{\partial^2 C}{\partial S^2} = \exp\Big(-\int_t^T \bar{r}(u)\,du\Big)\,\frac{\partial^2 U}{\partial S^2}.$$

Substitute these into the partial differential equation to get

$$\frac{\partial U}{\partial t} + \tfrac{1}{2}\bar{\sigma}(t)^2\,S^2\,\frac{\partial^2 U}{\partial S^2} + \big(\bar{r}(t) - \bar{y}(t)\big)\,S\,\frac{\partial U}{\partial S} = 0. \tag{7}$$

When $t = T$ we see that $\exp\Big(-\int_T^T \bar{r}(u)\,du\Big) = 1$ and so

$$U(S,T) = (S-K)^+. \tag{8}$$

Together with the Feynman Kăc result, (7) and (8) imply that

$$U(S,t) = \mathbb{E}_t\big[\,(S_T - K)^+ \,|\, S_t = S\,\big]$$

where S_t evolves as

$$\frac{dS_t}{S_t} = \big(\bar{r}(t) - \bar{y}(t)\big)\,dt + \bar{\sigma}(t)\,dW_t.$$

(c) Deduce that the solution to (6) is

$$S_T = S_t\,\exp\left(\int_t^T \big(\bar{r}(u) - \bar{y}(u) - \tfrac{1}{2}\bar{\sigma}(u)^2\big)du + \int_t^T \bar{\sigma}(u)\,dW_u\right).$$

One way to do this is to use the process $\log(S_t)$. Noting that

$$\frac{d\log(S)}{dS} = \frac{1}{S}, \quad \frac{d^2\log(S)}{dS^2} = -\frac{1}{S^2},$$

we find that

$$\begin{aligned} d\log(S_t) &= \frac{dS_t}{S_t} - \frac{1}{2}\,\frac{\bar{\sigma}(t)^2\,S_t^2\,dt}{S_t^2} \\ &= \big(\bar{r}(t) - \bar{y}(t) - \tfrac{1}{2}\bar{\sigma}(t)^2\big)\,dt + \bar{\sigma}(t)\,dW_t. \end{aligned}$$

Integrating from t to T gives the result.

(d) Hence deduce that for fixed $t < T$ the solution of (5) is

$$C(S,t) = C_{\rm bs}(S,t;K,T,\hat{r},\hat{y},\hat{\sigma})$$

where

$$\hat{r} = \frac{1}{T-t}\int_t^T \bar{r}(u)\,du,\ \hat{y} = \frac{1}{T-t}\int_t^T \bar{y}(u)\,du,\ \hat{\sigma}^2 = \frac{1}{T-t}\int_t^T \bar{\sigma}(u)^2\,du,$$

and $C_{\rm bs}(S,t;K,T,r,y,\sigma)$ is the Black-Scholes formula for a call with strike K, expiry T, constant risk-free rate r, constant continuous dividend yield y and constant volatility σ.

If $t < T$ is fixed then we can write

$$S_T = S_t\,\exp\left(\int_t^T \left(\bar{r}(u) - \bar{y}(u) - \tfrac{1}{2}\bar{\sigma}(u)^2\right)du + \int_t^T \bar{\sigma}(u)\,dW_u\right)$$

as

$$S_T = S_t\,\exp\left(\left(\hat{r} - \hat{y} - \tfrac{1}{2}\hat{\sigma}^2\right)(T-t) + \hat{\sigma}\,\hat{W}_\tau\right),$$

where $\tau = T - t$ and

$$\hat{r} = \frac{1}{T-t}\int_t^T \bar{r}(u)\,du,\ \hat{y} = \frac{1}{T-t}\int_t^T \bar{y}(u)\,du,\ \hat{\sigma}^2 = \frac{1}{T-t}\int_t^T \bar{\sigma}(u)^2\,du.$$

The final term, $\hat{\sigma}\,\hat{W}_\tau$, follows because the Itô integral

$$\int_t^T \bar{\sigma}(u)\,dW_u$$

is normally distributed with zero mean and variance given by

$$\int_t^T \bar{\sigma}(u)^2\,du.$$

Therefore, we can write

$$\int_t^T \bar{\sigma}(u)\,dW_u = \hat{\sigma}\,\hat{W}_\tau$$

where $\hat{W}_\tau$ is normally distributed with zero mean and variance $\tau = T - t$ and $\hat{\sigma}$ is as defined above.

Thus it follows that we could write

$$\begin{aligned}
C(S,t) &= e^{-\hat{r}(T-t)}\,\mathbb{E}_t\left[(S_T - K)^+ \,|\, S_t = S,\ dS_t/S_t = \left(\bar{r}(t) - \bar{y}(t)\right)dt + \sigma(t)\,dW_t\right] \\
&= e^{-\hat{r}(T-t)}\,\mathbb{E}_t\left[(S_T - K)^+ \,|\, S_T = S\,e^{(\hat{r}-\hat{y}-\frac{1}{2}\hat{\sigma}^2)(T-t)+\hat{\sigma}\hat{W}_{T-t}}\right] \\
&= C_{\rm bs}(S,t;K,T,\hat{r},\hat{y}\,\hat{\sigma}),
\end{aligned}$$

where $C_{\rm bs}(S,t;K,T,\hat{r},\hat{y}\,\hat{\sigma})$ is the Black-Scholes formula for the price of a call option with constant risk-free rate $\hat{r}$, constant continuous dividend yield $\hat{y}$ and constant volatility $\hat{\sigma}$.

10. A European log-put option has the payoff

$$V_T = \big(-\log(S_T/K)\big)^+$$

(a) Show that if S_u evolves as

$$\frac{dS_u}{S_u} = r\,du + \sigma\,dW_u, \;\; t < u \le T, \quad S_t = S,$$

then

$$\text{prob}(S_T < K) = \mathrm{N}(-d_-), \quad d_- = \frac{\log(S/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sqrt{\sigma^2\,(T-t)}}.$$

As $S_t = S$, we can write

$$S_T = S\,e^{(r-\sigma^2/2)\tau + \sigma\,W_\tau}, \quad \tau = T - t,$$

from which we can see that

$$\begin{aligned}
\text{prob}(S_T < K) &= \text{prob}\big(\log(S_T) < \log(K)\big) \\
&= \text{prob}\big(\log(S) + (r - \tfrac{1}{2}\sigma^2)\tau + \sigma\,W_\tau < \log(K)\big) \\
&= \text{prob}\big(\sigma\,W_\tau < -\log(S/K) - (r - \tfrac{1}{2}\sigma^2)\tau\big) \\
&= \text{prob}\Big(\big(W_\tau/\sqrt{\tau}\big) < -\frac{\log(S/K) + (r - \frac{1}{2}\sigma^2)\tau}{\sqrt{\sigma^2\,\tau}}\Big) \\
&= \mathrm{N}(-d_-),
\end{aligned}$$

because $(W_\tau/\sqrt{\tau}) \sim N(0,1)$.

(b) Assuming the underlying share pays no dividends, show that the Black-Scholes value function for the log-put is

$$V(S,t) = e^{-r(T-t)}\sqrt{\sigma^2\,(T-t)}\,\Big(d_-\,\mathrm{N}(-d_-) - e^{-\frac{1}{2}d_-^2}/\sqrt{2\pi}\Big).$$

The simplest way to do this is to use the formula

$$V(S,t) = e^{-r(T-t)}\,\mathbb{E}_t^{\mathbb{Q}}\Big[\,\big(-\log(S_T/K)\big)^+ \,\big|\, S_t = S\Big],$$

where for $t < u$

$$\frac{dS_u}{S_u} = r\,du + \sigma\,dW_u, \quad S_t = S.$$

With $\tau = T - t$, we have

$$\begin{aligned}
S_T &= S\,\exp\big((r - \tfrac{1}{2}\sigma^2)(T-t) + \sigma\,W_\tau\big) \\
&= S\,\exp\big((r - \tfrac{1}{2}\sigma^2)(T-t) + \sqrt{\sigma^2\,\tau}\,Z\big),
\end{aligned}$$

where $Z = W_\tau/\sqrt{\tau} \sim N(0,1)$, and so

$$\log(S_T/K) = \log(S/K) + (r - \tfrac{1}{2}\sigma^2)\tau + \sqrt{\sigma^2\,\tau}\,Z. \tag{9}$$

Following the same idea as in (a), $-\log(S_T/K) > 0$ iff and only if

$$\begin{aligned} & 0 < -\log(S/K) + (r - \tfrac{1}{2})\tau - \sqrt{\sigma^2\,\tau}\,Z \\ \iff \quad & \sqrt{\sigma^2\,\tau}\,Z < -\log(S/K) + (r - \tfrac{1}{2})\tau \\ \iff \quad & Z < -\frac{\log(S/K) + (r - \frac{1}{2})\tau}{\sqrt{\sigma^2\,\tau}} \\ \iff \quad & Z < -d_-. \end{aligned}$$

Regarding $\log(S_T/K)$ as a function of the random variable $Z \sim \mathcal{N}(0,1)$, as in (9), we see that

$$\begin{aligned} & \mathbb{E}_t^{\mathbb{Q}}\big[\big(-\log(S_T/K)\big)^+ \,\big|\, S_t = S\big] \\ & = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \big(-\log(S_T/K)\big)^+ e^{-z^2/2}\,dz \\ & = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-d_-} \log(S_T/K)\,e^{-z^2/2}\,dz. \end{aligned}$$

Using (9) to express $\log(S_T/K)$ in terms of Z, we get

$$\begin{aligned} & \mathbb{E}_t^{\mathbb{Q}}\Big[\big(-\log(S_T/K)\big)^+ \,\big|\, S_t = S\Big] \\ & = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-d_-} \big(\log(S/K) + (r - \tfrac{1}{2}\sigma^2)\tau + \sqrt{\sigma^2\,\tau}\,z\big)\,e^{-\frac{1}{2}z^2}\,dz \\ & = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-d_-} \sqrt{\sigma^2\tau}\,\big(d_- + z\big)\,e^{-\frac{1}{2}z^2}\,dz \\ & = -\sqrt{\sigma^2\tau}\,\Big(d_-\,\mathrm{N}(-d_-) - \frac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}z^2}\Big|_{-\infty}^{-d_-}\Big) \\ & = \sqrt{\sigma^2(T-t)}\,\Big(\frac{e^{-\frac{1}{2}d_-^2}}{\sqrt{2\pi}} - d_-\,\mathrm{N}(-d_-)\Big). \end{aligned}$$

Multiplying this by $e^{-r(T-t)}$ gives the result.

11. Let $0 < T_1 < T_2$ and $K > 0$. A derivative security with the following properties is written on a share (which does not pay any dividends between time $t = 0$ and $t = T_2$. If at time T_1 the share price is greater than or equal to K, $S_{T_1} \geq K$, then the derivative security becomes a European call option with strike S_{T_1} and expiry date T_2.

If $S_{T_1} < K$, it becomes a European put option with strike S_{T_1} and expiry date T_2. Find the Black-Scholes price function for this security when $T_1 < t < T_2$ and then when $0 \le t \le T_1$.

For $T_1 \le t \le T_2$ we know the value of S_{T_1}. If $S_{T_1} \ge K$ then we have a call option with strike S_{T_1} so

$$V(S,t) = C_{\mathrm{bs}}(S,t;\mathrm{strike} = S_{T_1})$$

while if $S_{T_1} < K$ then we have a put option with strike S_{T_1}, so

$$V(S,t) = P_{\mathrm{bs}}(S,t;\mathrm{strike} = S_{T_1}).$$

Thus for $T_1 \le t \le T_2$ we have

$$V(S,t) = \begin{cases} S_{T_1}\, e^{-r(T_2-t)}\, \mathrm{N}(-d_-) - S\,\mathrm{N}(-d_+) & \text{if } S_{T_1} < K, \\ S\,\mathrm{N}(d_+) - S_{T_1}\, e^{-r(T_2-t)}\, \mathrm{N}(d_-) & \text{if } S_{T_1} \ge K, \end{cases}$$

where

$$d_\pm = \frac{\log(S/S_{T_1}) + (r \pm \frac{1}{2}\sigma^2)(T_2 - t)}{\sqrt{\sigma^2(T_2-t)}}.$$

At time T_1, by definition $S = S_{T_1}$ so

$$V(S,T_1) = \begin{cases} A(T_1,T_2)\,S & \text{if } S < K, \\ B(T_1,T_2)\,S & \text{if } S \ge K \end{cases}$$

where

$$\begin{aligned} A(T_1,T_2) &= e^{-r(T_2-T_1)}\,\mathrm{N}(-\hat{d}_-) - \mathrm{N}(-\hat{d}_+) \\ B(T_1,T_2) &= \mathrm{N}(\hat{d}_+) - e^{-r(T_2-T_1)}\,\mathrm{N}(\hat{d}_-) \end{aligned}$$

and

$$\hat{d}_\pm = \frac{(r \pm \frac{1}{2}\sigma^2)(T_2 - T_1)}{\sqrt{\sigma^2(T_2-T_1)}}.$$

If we use the fact that $\mathrm{N}(x) + \mathrm{N}(-x) = 1$, we see that

$$\begin{aligned} &B(T_1,T_2) - A(T_1,T_2) \\ &\qquad = \mathrm{N}(\hat{d}_+) + \mathrm{N}(-\hat{d}_+) - e^{-r(T_2-T_1)}\big(\mathrm{N}(\hat{d}_-) + \mathrm{N}(-\hat{d}_-)\big) \\ &\qquad = 1 - e^{-r(T_2-T_1)}, \end{aligned}$$

which establishes that

$$B(T_1,T_2) = A(T_1,T_2) + C(T_1,T_2)\,\mathbf{1}_{\{S \ge K\}},$$

where $C(T_1,T_2) = 1 - e^{-r(T_2-T_1)}$. Thus we can write

$$V(S,T_1) = A(T_1,T_2)\,S + C(T_1,T_2)\,S\,\mathbf{1}_{\{S<K\}}.$$

As the share pays no dividends, any multiple of S is a solution of the Black-Scholes equation and so the component $A(T_1, T_2)\, S$ of the payoff leads to a price that is always $A(T_1, T_2)\, S$. The component of the payoff $C(T_1, T_2)\, S\, \mathbf{1}_{\{S \geq K\}}$ is simply $C(T_1, T_2)$ gap-calls — $S\, \mathbf{1}_{\{S \geq K\}}$ is zero if $S < K$ and S if $S \geq K$. Thus if we denote the price function for a gap-call with strike K by $C_g(S, t; K, T_1)$ we find that for $t < T_1$ we have

$$V(S, t) = A(T_1, T_2)\, S + C(T_1, T_2)\, C_g(S, t; K, T_1).$$

Although you were not asked to find a formula for $C_g(S, t)$, it is a relatively simple thing to do. Recall from lectures that:

- if $U(S, t)$ is a solution of the Black-Scholes equation then so too is $S\,(\partial U/\partial S)$;
- the delta of a call option is given by $\Delta(S, t) = (\partial C/\partial S)(S, t) = \mathrm{N}(d_+)$; and
- $\Delta_c(S, t)$ has the property that $\Delta_c(S, T) = \mathbf{1}_{\{S \geq K\}}$.

It follows that $S\, \Delta_c(S, t) = S\, \mathrm{N}(d_+)$ is a solution of the Black-Scholes equation with the property that $S\, \Delta_c(S, T) = S\, \mathbf{1}_{\{S \geq K\}}$ and so we must have

$$C_g(S, t; K, T) = S\, \mathrm{N}(d_+),$$

where, as usual,

$$d_+ = \frac{\log(S/K) + (r + \frac{1}{2}\sigma^2)(T - t)}{\sqrt{\sigma^2(T - t)}}.$$

12. Assume that the USD/GBP exchange rate, X_t, evolves according to the SDE

$$\frac{dX_t}{X_t} = \mu\, dt + \sigma\, dW_t.$$

(a) Given that today's exchange rate, at $t = 0$, is X_0 find the expected USD/GBP exchange rate $\mathbb{E}[X_T]$ at time $T > 0$.

Write $dX_t = \mu\, X_t\, dt + \sigma\, X_t\, dW_t$ and integrate

$$X_t - X_0 = \mu \int_0^t X_u\, du + \sigma \int_0^T X_u\, dW_u$$

then take expectations to get

$$\begin{aligned} \mathbb{E}\big[X_t\big] - X_0 &= \mu\, \mathbb{E}\Big[\int_0^t X_u\, du\Big] + \sigma\, \mathbb{E}\Big[\int_0^t X_u\, dW_u\Big] \\ &= \mu \int_0^t \mathbb{E}[X_u]\, du \end{aligned}$$

and then differentiate with respect to t,

$$\frac{d\mathbb{E}\big[X_t\big]}{dt} = \mu\,\mathbb{E}\big[X_t\big].$$

Solve this for $\mathbb{E}\big[X_t\big]$ to find

$$\mathbb{E}\big[X_t\big] = X_0\,e^{\mu t},$$

which gives $\mathbb{E}\big[X_T\big] = X_0\,e^{\mu T}$.

(b) Find the SDE which the GBP/USD exchange rate, $Y_t = 1/X_t$ follows.

Applying Itô's lemma to $Y_t = f(X_t)$ where $f(X) = 1/X$ gives

$$\begin{aligned} dY_t &= -\frac{dX_t}{X_t^2} + \frac{d[X]_t}{X_t^3} \\ &= -\frac{\mu}{X_t}\,dt - \frac{\sigma}{X_t}\,dW_t + +\frac{\sigma^2}{X_t}\,dt \\ &= (\sigma^2-\mu)\,Y_t\,dt - \sigma\,Y_t\,dW_t, \end{aligned}$$

that is,

$$\frac{dY_t}{Y_t} = (\sigma^2-\mu)\,dt - \sigma\,dW_t.$$

(c) Given that $Y_0 = 1/X_0$ today, find the expected GBP/USD exchange rate $\mathbb{E}[Y_T]$ at time $T > 0$.

The SDE satisfied by Y_t has the same form as that for X_t but with μ replaced by $\sigma^2 - \mu$ and σ replaced by $-\sigma$. Therefore

$$\mathbb{E}[Y_T] = Y_0\,e^{(\sigma^2-\mu)T} = \frac{1}{X_0}\,e^{(\sigma^2-\mu)T}.$$

(d) Show that, although $X_T\,Y_T = 1$ for any $T > 0$,

$$\mathbb{E}[X_T]\,\mathbb{E}[Y_T] = e^{\sigma^2 T}.$$

By definition, $Y_T = 1/X_T$ so $X_T\,Y_T = 1$. From the calculations above,

$$\mathbb{E}[X_T]\,\mathbb{E}[Y_T] = e^{\sigma^2 T},$$

which is strictly greater than one unless $T = 0$ or $\sigma = 0$.

13. An investor has the choice of investing their wealth of 1 unit of currency in either a risky asset whose price evolves as

$$\frac{dS_t}{S_t} = \mu\,dt + \sigma\,dW_t,\ \ t > 0, \quad S_0 = 1,$$

where $\sigma > 0$, or in a risk-free bond whose price evolves as

$$\frac{dB_t}{B_t} = r\,dt, \;\; t > 0, \quad B_0 = 1,$$

where $0 < r < \mu - \frac{1}{2}\sigma^2$. The investment horizon is $[0, T]$. The investor decides to invest their funds in the risky asset, but is worried that when they withdraw the funds, at time T, the risk-free bonds may have outperformed the risky assets. So they consider the possibility of purchasing a put option with maturity T to protect themselves against this possibility. (They borrow money to buy the put.)

(a) What is the probability of the risky asset underperforming the risk-free one, i.e, what is the probability that $S_T < e^{rT}$?

As we start with $S_0 = 1$ we have

$$S_T = e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma W_T}$$

and we want to know

$$\begin{aligned}
\text{prob}\big(\, S_T < e^{rT} \,\big) \;&=\; \text{prob}\big(\, (\mu - \tfrac{1}{2}\sigma^2)T + \sigma W_T < rT \,\big) \\
&=\; \text{prob}\big(\, \sigma W_T < (r - \mu + \tfrac{1}{2}\sigma^2)T \,\big) \\
&=\; \text{prob}\Big(\, \frac{W_T}{\sqrt{T}} < \frac{(r - \mu + \frac{1}{2}\sigma^2)T}{\sqrt{\sigma^2\, T}} \,\Big) \\
&=\; \text{N}(\, x \,),
\end{aligned}$$

since $W_T/\sqrt{T} \sim \mathcal{N}(0, 1)$ and where

$$x = \frac{(r - \mu + \frac{1}{2}\sigma^2)\sqrt{T}}{\sigma}.$$

(b) What happens to this probability as $T \to \infty$?

We have $r + \frac{1}{2}\sigma^2 < \mu$ and $\sigma > 0$, so

$$\frac{(r - \mu + \frac{1}{2}\sigma^2)}{\sigma} < 0$$

and so $x \to -\infty$ as $T \to \infty$. This means the probability of underperformance goes to zero as $T \to \infty$.[4]

[4]It goes to zero extremely rapidly. To see this consider

$$\int_{-\infty}^{-z} e^{-y^2/2}\, dy = \int_z^{\infty} e^{-y^2/2}\, dy = \int_z^{\infty} y\, e^{-y^2/2}\, \frac{dy}{y} = \frac{e^{-z^2/2}}{z} - \int_z^{\infty} e^{-y^2/2}\, \frac{dy}{y^2}$$

which shows that as $z \to \infty$, $\text{N}(-z) \sim \dfrac{e^{-z^2/2}}{\sqrt{2\pi}\; z}$.

(c) What should the strike of the put be in order that the investor is completely insured against the possibility of underperformance?

$K = e^{rT}$. Then at T if $S_T < e^{rT}$ the investor can sell the risky asset for e^{rT} by exercising the put and if $S_T \geq e^{rT}$ they can let the put expire worthless. Thus they are guaranteed a final value of $\max(e^{rT}, S_T)$.

(d) What happens to the price of the insurance as $T \to \infty$?

As $S_0 = 1$ and $K = e^{rT}$, the price of the put at $t = 0$ is

$$\begin{aligned} P(1, 0; T) &= e^{rT} e^{-rT} \,\mathrm{N}(-d_-) - (\,-d_+) \\ &= \mathrm{N}(-d_-) - \mathrm{N}(-d_+) \end{aligned}$$

with

$$d_\pm = \frac{\log(1/e^{rT}) + (r \pm \frac{1}{2}\sigma^2)T}{\sqrt{\sigma^2\, T}} = \frac{\pm\frac{1}{2}\sigma^2\, T}{\sqrt{\sigma^2\, T}} = \pm\tfrac{1}{2}\sqrt{\sigma^2\, T}.$$

This shows that

$$\begin{aligned} P(1, 0; T) &= \mathrm{N}\big(\sqrt{\sigma^2\, T}\,\big) - \mathrm{N}\big(-\sqrt{\sigma^2\, T}\,\big) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{\sigma^2 T}}^{\sqrt{\sigma^2 T}} e^{-p^2/2}\, dp \leq 1, \end{aligned} \tag{10}$$

and it follows that

$$\lim_{T\to\infty} P(1, 0; T) = 1,$$

i.e., the cost of insurance against underperformance tends to the total value available for investment, even though the probability of underperformance tends to zero.

More generally, it is clear that the risk of underperformance is monotonically decreasing in T but the cost of insurance against underperformance, i.e, the put, is an increasing function of T. If the investor borrows the money to buy the put at $t = 0$, they will owe $e^{rT} P(1, 0; T)$ at time T and they are guaranteed to have $\max(e^{rT}, S_T)$ and so their overall position at time T is

$$\max\Big(\big(1 - P(1, 0; T)\big)\, e^{rT}, S_T - e^{rT} P(1, 0; T)\Big) \geq 0.$$