B8.3 Week 1 Summary 2019

1 Assumptions

1. There is a riskless investment (a bank account or bonds) which grows at a constant, continuously compounded rate r. If M_t is invested at time t then it grows to $M_T = M_t\, e^{r(T-t)}$ at time $T > t$. A guaranteed amount of B_T paid at time T is worth $B_t = B_T\, e^{-r(T-t)}$ at time $t < T$. Borrowing and lending rates are both assumed equal to r.

2. There are no trading costs; if an asset can be bought for S_t at time t it can be sold for S_t at time t.

3. An arbitrage is an investment which costs nothing (or less) to set up at time t, $X_t \leq 0$, but at a later time $T > t$ has:

   (a) zero probability of having a negative value, $\text{prob}(X_T < 0) = 0$;

   (b) strictly non-zero probability of having a strictly positive value, $\text{prob}(X_T > 0) > 0$.

   We assume that *no arbitrage opportunities exist.* (In practice they do, but when institutions exploit them supply and demand causes prices to realign in order to eliminate them.)

4. Short-selling is allowed. This is *often* true.

5. Assets are infinitely divisible, so it possible to own 0.432 shares for instance. This is not a major issue as forwards, calls and puts are usually written on $1,000$s or $10,000$s of shares, rather than one share.

2 Products

For simplicity, the following products are written on a share, with price $S_\tau > 0$, which pays no dividends, and costs nothing to hold, between times t and $T > t$.

2.1 Forwards

A forward is an agreement entered into by two parties at time t in which the holder (who has the long position) promises to pay the agreed forward price $F_t > 0$ for the share at some given maturity date $T > t$, and the writer (who has the short position) promises to deliver the share at time T for the forward price F_t. Both parties are obliged to go through with the transaction regardless of the share price, $S_T > 0$, at maturity. Under normal circumstances, neither party has to pay to enter the agreement at time t.

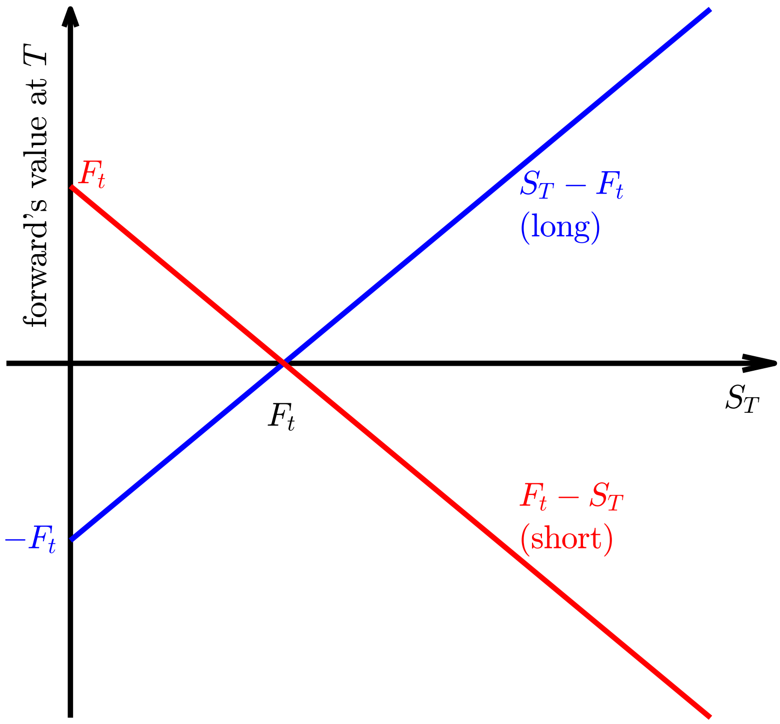

Figure 1: Payoff diagrams for long and short forward positions

For the short position, the forward may be hedged by borrowing the price of the share at time t, S_t, buying the share, holding it to maturity then delivering it in return for F_t. For the long position, it may be hedged by short selling the share at t, putting the money in a risk-free account and then using the forward to buy back the share for F_t and close out the short sale. If there is no arbitrage then

$$F_t = S_t\, e^{r(T-t)}.$$

The payoff diagram for the forward for the long position is a plot of the value of the forward to the holder at maturity against the value of the share at maturity which is $S_T - F_t$.

2.2 Call options

A call option is a contract with an expiry date $T > t$ and a strike $K > 0$ in which:

1. the holder (who has the long position) has the *right* to buy the underlying share for the strike at the expiry date;

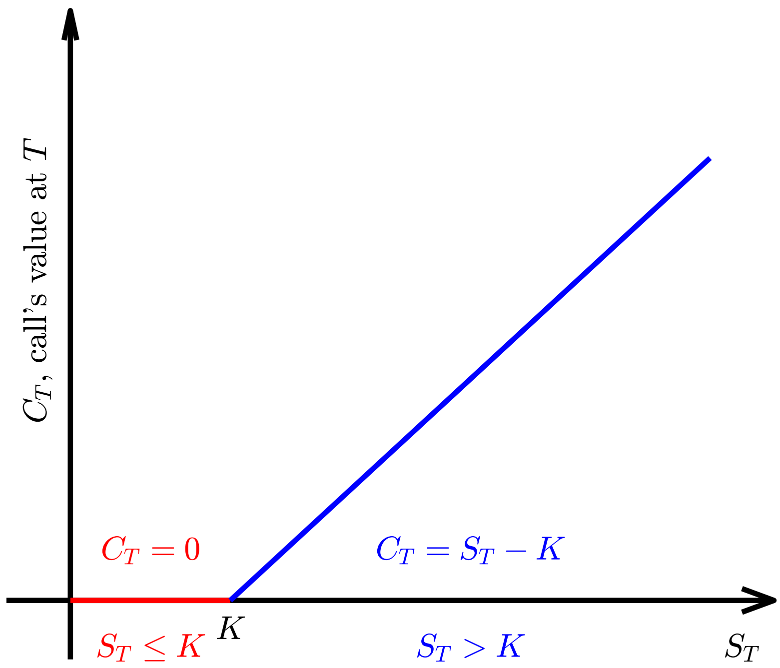

Figure 2: Payoff diagram for a long call position

2. the writer (who has the short position) is *obliged* to deliver the share for the strike if the holder exercises their right.

The value (of the long position) of the call at expiry is

$$\max(S_T - K, 0) = (S_T - K)^+$$

and a plot of this function is the call's payoff diagram. Unlike forwards, the holder has to pay a positive amount for the call option (this is the consequence of no arbitrage).

2.3 Put options

A put option is a contract with an expiry date $T > t$ and a strike $K > 0$ in which

1. the holder (who has the long position) has the *right* to sell the underlying share for the strike at the expiry date;

2. the writer (who has the short position) is *obliged* to buy the share for the strike if the holder exercises their right.

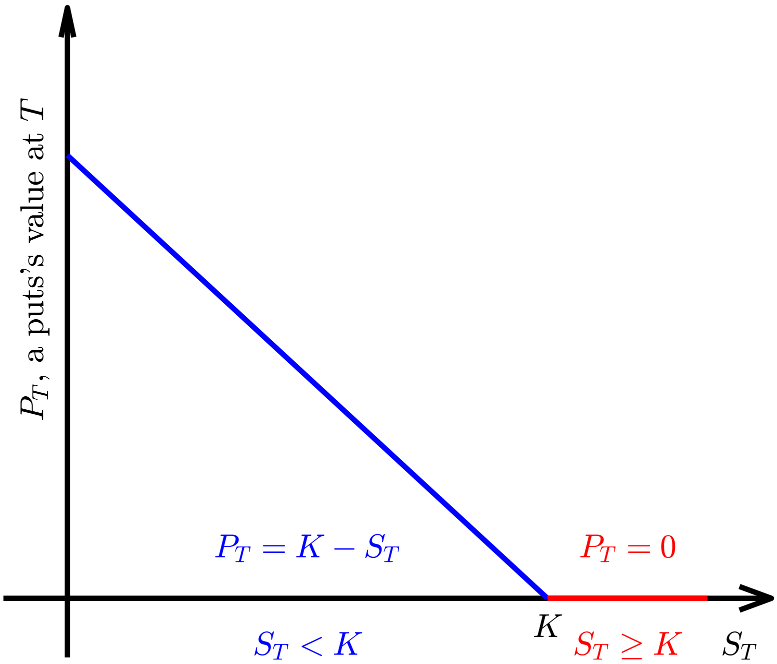

Figure 3: Payoff diagram for a long put position

The value (of the long position) of the call at expiry is

$$\max(K - S_T, 0) = (K - S_T)^+$$

and a plot of this function is the put's payoff diagram. The holder has to pay a positive amount for a put option (this is the consequence of a no arbitrage argument).

3 European vs American option

An option which may be exercised *only* at its expiry date is called a European option, one which may be exercised at any time up to and including its expiry date is called an American option. Options will be assumed European unless stated otherwise.

4 Put-call parity

Let $c(S_t, t; K, T)$ be the price at time t of a European call option with strike K and expiry T and $p(S_t, t; K, T)$ be the price of a European put on the same share and with the same strike and expiry. If a portfolio has one long call and one short put its value at t is

$$c(S_t, t; K, T) - p(S_t, t; K, T)$$

and at expiry its value is

$$(S_T - K)^+ - (K - S_T)^+ = S_T - K.$$

Consider a portfolio consisting at time t of the asset and a loan of $K\,e^{-r(T-t)}$. At expiry this portfolio also has value $S_T - K$. As the two portfolios also have identical cash flows over $(t, T]$, no arbitrage implies they must have the same value at time t;

$$c(S_t, t; K, T) - p(S_t, t; K, T) = S_t - K\,e^{-r(T-t)}$$

This is known as put-call parity.

5 Law of one price

Put-call parity is a special case of the law of one price; if two portfolios have identical cash flows over the time interval $(t, T]$ and are guaranteed to have the same values at time T, under all possible circumstances, then they must have the same value at time t. If this were not the case, there would be an arbitrage opportunity (at t, short sell the expensive one and buy the cheaper one then close the position out at time T).