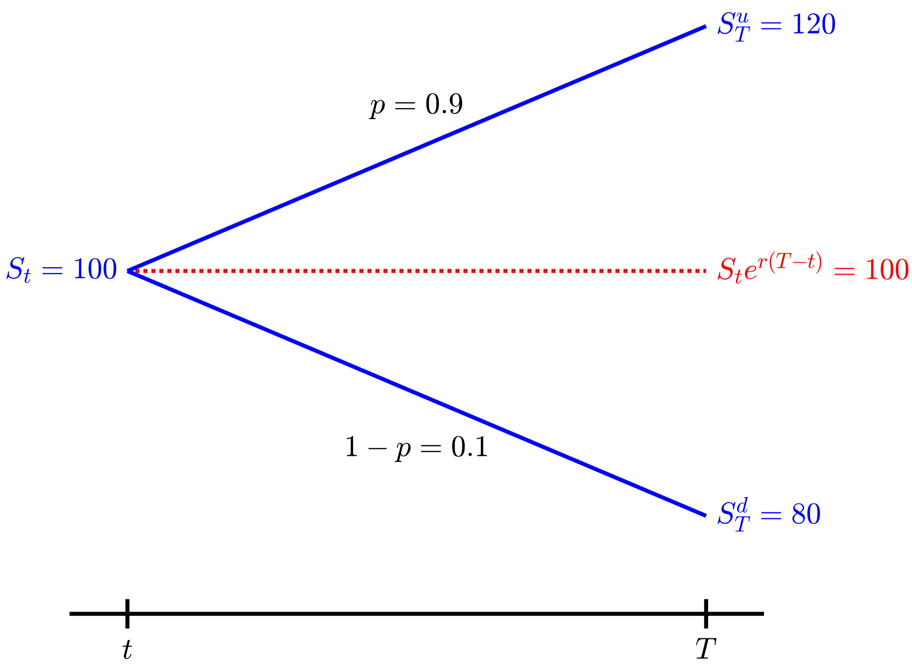

Figure 1: Underlying share prices in this model

B8.3 Week 2 2019: A specific example

Consider a one-step binomial model where the current share price is $S_t = 100$ and where the price at expiry, $T > t$, is either $S_T^u = 120$ or $S_T^d = 80$. Suppose that the 'real' probability of the share price going up to S_T^u is $p = 0.9$ and the 'real' probability of the share price going down to S_T^d is $1 - p = 0.1$.

For simplicity, take the risk-free rate (i.e., the interest rate) to be zero,

$$r = 0,$$

so we have $S_t\, e^{r(T-t)} = S_t = 100$. This clearly satisfies the inequalities

$$S_T^d = 80 < S_t\, e^{r(T-t)} = 100 < S_T^u = 120,$$

so this share price model is arbitrage free.

Suppose that a European call option is written on the share, with strike $K = 100$ and expiry T. If the share price goes up to $S_T^u = 120$ then the call is worth $C_T^u = 20$ and if the share price goes down to $S_T^d = 80$ the call price is $C_T^d = 0$. That is,

$$C_T = \begin{cases} C_T^d = 0 & \text{if } S_T^d = 80 \text{ with prob} = 0.1, \\ C_T^u = 20 & \text{if } S_T^u = 120 \text{ with prob} = 0.9. \end{cases}$$

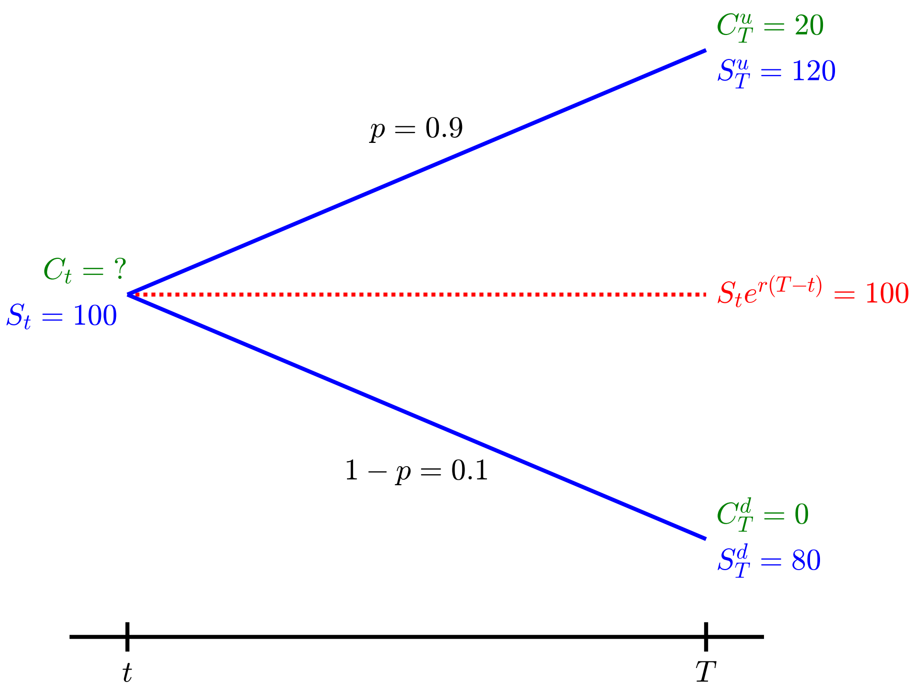

Figure 2: Underlying share and call prices in this model

A naive price for the call

Suppose we decide to price the option using the given probabilities p and $(1-p)$ (and then trivially discount this back to the present using the zero interest rate) using the formula

$$\begin{aligned}\hat{C}_t &= e^{-r(T-t)}\,\mathbb{E}^p[\,C_T\,] \\ &= p\,C_T^u + (1-p)\,C_T^d \\ &= 0.9 \times 20 + 0.1 \times 0 \\ &= 18.\end{aligned}$$

As we shall see, this price can be arbitraged and so is *not* the correct price.

Delta hedging argument

At time t set up a portfolio Π long an option and short Δ_t shares

$$\Pi_t = C_t - \Delta_t\,S_t,$$

and hold this portfolio fixed until time T. Choose Δ_t so that the portfolio has the same value regardless of whether the up-state or the down-state occurs, $C_T^d - \Delta_t\,S_T^d = C_T^u - \Delta_t\,S_T^u$. This gives

$$\Delta_t = \left(\frac{C_T^u - C_T^d}{S_T^u - S_T^d}\right) = \left(\frac{20-0}{120-80}\right) = \frac{1}{2}.$$

This portfolio is risk-free and so must grow at the risk-free rate (which is zero in this case) or there would be an arbitrage opportunity, which implies that

$$C_t - \Delta_t\, S_t = C_T^u - \Delta_t\, S_T^u.$$

Putting in the numbers, this becomes

$$\begin{aligned} C_t - \tfrac{1}{2} \times 100 &= 20 - \tfrac{1}{2} \times 120 \\ C_t - 50 &= 20 - 60 \\ C_t &= 10. \end{aligned}$$

Now suppose that someone is willing to pay you $\hat{C}_t = 18$ for the call at time t. You should take the price $\hat{C}_t = 18$ and write the call for them. At time t, take the delta-hedging price or *fair price* of $C_t = 10$ from the $\hat{C}_t = 18$ and put the remaining $\hat{C}_t - C_t = 8$ in the bank. According the to delta-hedging calculation above, you have to buy $\Delta_t = \frac{1}{2}$ a share at time t, which will cost you $\frac{1}{2} \times 100 = 50$. Given that you have the fair price of 10, you will have to borrow 40 in order to do this. So at time t your position is

written a call with strike $K = 100$ and expiry T,
borrowed 40,
bought half a share,
you have the excess 8 in another bank account.

Sit on this position until expiry T. At expiry, T, as the interest rate is zero you still owe 40 and the value of the excess 8 in the bank is still 8. You still have $\frac{1}{2}$ a share and you must pay out on the option if it is exercised.

If the share price has gone up the call is worth 20 and the share is worth 120. Sell your $\frac{1}{2}$-a-share for 60. The holder of the call will pay you 100 to exercise the call, you add another 20 (from the 60) and buy the share and give it to them.[1] This leaves you with 40 in cash and you use this to pay off the loan of 40. You still have the excess 8 in the bank.

If the share price has gone down to 80 then the call is worthless and should not be exercised.[2] You have half a share which you can sell for 40 and repay the loan of 40. You still have the excess $\hat{C}_t - C_t = 8$ in the bank.

Thus, whether the share price goes up or down, you still have $\hat{C}_t - C_t = 8$ in the bank. Your net investment in this scheme is zero and hence you have just arbitraged the person who paid you $\hat{C}_t = 18$ for the call option.

[1] You could agree with the holder of the call to pay them the 20 directly, rather than them giving you 100 and you giving them the share worth 120—this is called cash-settlement, as opposed to physical settlement described above. Either way, your net pay out at this step is still 20.

[2] If the holder is silly enough to exercise the call option, which might be possible given they've already paid you 18 for the call which is only worth 10, then they give you the strike, 100, you take 80 of this and buy the share and give it to them. Put the other 20 in with the 8 already in the bank thus *increasing* your arbitrage profit.

Clearly you can play this game with anyone who is prepared to pay you strictly more than the fair price of $C_t = 10$. If someone believes the price should be strictly less than the fair price of $C_t = 10$, *buy* the call from them (instead of writing the call for them) and reverse all the other positions described above—this will also result in an arbitrage.

Self-financing replication argument

At time t set up a portfolio whose value is Φ_t with ϕ_t shares and ψ_t bonds (bonds grow at the risk-free rate, which in this case is 1)

$$\Phi_t = \phi_t \, S_t + \psi_t.$$

With the model given above, this becomes

$$\Phi_t = 100 \, \phi_t + \psi_t,$$

assuming the bond price is 1 at time t. Hold this portfolio fixed and choose ϕ_t and ψ_t so that the portfolio has value $C_T^u = 20$ if the share price goes up to $S_T^u = 120$ and $C_T^d = 0$ if the share price goes down to $S_T^d = 80$,

$$\begin{aligned} \Phi_T^u &= 120 \, \phi_t + \psi_t &= 20, \\ \Phi_T^d &= 80 \, \phi_t + \psi_t &= 0. \end{aligned}$$

Solving for ϕ_t, by subtracting the second equation from the first, gives

$$\phi_t = \frac{20}{40} = \tfrac{1}{2}.$$

It then follows from either of these equations that

$$\psi_t = 20 - \tfrac{1}{2} \times 120 = 0 - \tfrac{1}{2} \times 80 = -40.$$

What these calculations show is that if I buy half a share at time t, costing 50, and borrow 40 (in bonds) then I will perfectly replicate[3] the payoff of the option at time T. By a simple no-arbitrage argument, the cost of the call must be the difference, $C_t = \tfrac{1}{2} \, 100 - 40 = 50 - 40 = 10$. Put another way, you pay me 10 for the option, I borrow 40 and then buy the half the share for $10 + 40 = 50$. This costs me nothing. At expiry I have exactly the payoff of the option whatever happens.

Now suppose that you believe the correct price for the option is $\hat{C}_t = 18$. I will sell you the option for 18, put 8 in a bank account, borrow 40 and add this to the remaining 10, to give 50, and buy half a share at time t. My net outlay is zero to do this (including the 8 as I've banked this). I then sit on the position until expiry, T.

[3] In the sense that I will have exactly $C_T^u = 20$ is the share price goes up to $S_T^u = 120$ and if the share price goes down to $S_T^d = 80$ I will have exactly $C_T^d = 0$.

If the share price has gone up to $S_T^u = 120$ then, excluding the 8 excess in the bank, my position is

$$\Phi_T^u = \tfrac{1}{2}\, S_T^u - 40 = \tfrac{1}{2}\, 120 - 40 = 20 = C_T^u$$

and similarly if the share price goes down to $S_T^d = 80$ then the value of my position is

$$\Phi_T^d = \tfrac{1}{2}\, S_T^d - 40 = \tfrac{1}{2}\, 80 - 40 = 0 = C_T^d.$$

Thus, whether the share price goes up or down, I am perfectly covered at expiry.[4] Moreover, I still have the 8 in the bank. In effect, by paying me $\hat{C}_t = 18$ for the option you have given me $\hat{C}_t - C_t = 8$ for free, which is an arbitrage. The same argument works for *any* price $\hat{C}_t$ strictly greater than the fair price of $C_t = 10$.

If you believe the fair price is less than 10 then I *buy* the call from you (rather than writing it) and reverse the strategy given above. This also results in an arbitrage for me (i.e., a guaranteed (positive) profit for me and a guaranteed loss for you).

An observation

You might note that nowhere did I use the risk-neutral probabilities in these arguments, only the concept of delta-hedging and/or replication. The risk-neutral probabilities are simply a way of writing the fair price, C_t, in terms of $e^{r(T-t)}$ (which is 1 in this case) and C_T^d and C_T^u. As shown above, you can compute the fair price C_t without explicitly computing the risk-neutral probabilities.

[4]Specifically, if the share price goes up to $S_T^u = 120$ the call will be exercised. The holder will give me the strike, $K = 100$. My portfolio is worth 20 and so (excluding the extra 8 in a different bank account) I have exactly 120 in cash. I use this to buy the share for $S_T^u = 120$ and give it to the holder. If the share price goes down then the call will not be exercised, I don't have to deliver the share and the holder will pay me nothing. My portfolio is worth nothing in this case and so (excluding the excess 8 in another bank account) my overall position is zero.