B8.3 Week 6 summary 2019

European calls and puts

The Black-Scholes value of a European call option is

$$\begin{aligned} C(S,t) &= e^{-r(T-t)}\,\mathbb{E}_t\big[\,(S_T-K)^+\,|\,S_t=S\,\big] \\ &= S\,e^{-y(T-t)}\,\mathrm{N}(d_+) - K\,e^{-r(T-t)}\,\mathrm{N}(d_-) \end{aligned}$$

where in the expectation in the first line we have

$$\frac{dS_t}{S_t} = (r-y)\,dt + \sigma\,dW_t \tag{1}$$

and in the second line

$$d_\pm = \frac{\log(S/K) + (r - y \pm \frac{1}{2}\sigma^2)(T-t)}{\sqrt{\sigma^2(T-t)}}.$$

Note that r represents that *same* risk-free rate in *all* of these expressions; the risk-free rate is a property of bank accounts and/or bond prices. The delta of the call (the number of shares we hold to be perfectly hedged) is

$$\Delta_c(S,t) = e^{-y(T-t)}\,\mathrm{N}(d_+).$$

Consider the limit $t \to T^-$. In this limit we have

$$\lim_{t\to T^-} d_\pm = \lim_{t\to T^-} \frac{\log(S/K)}{\sqrt{\sigma^2(T-t)}} \to \begin{cases} -\infty & \text{if } 0<S<K, \\ 0 & \text{if } 0<S=K, \\ \infty & \text{if } 0<K<S \end{cases}$$

which implies that

$$\begin{aligned} \lim_{t\to T^-} C(S,t) &= \begin{cases} 0 & \text{if } 0<S\le K, \\ S-K & \text{if } 0<K<S, \end{cases} \\ \lim_{t\to T^-} \Delta_c(S,t) &= \begin{cases} 0 & \text{if } 0<S<K, \\ 1 & \text{if } 0<K<S, \end{cases} \end{aligned}$$

which shows that the option does indeed replicate the payoff and the delta is zero if the option isn't going to be exercised but is one if the option is going to be exercised.

Put-call parity,

$$C(S,t;K,T) - P(S,t;K,T) = S\,e^{-y(T-t)} - K\,e^{-r(T-t)},$$

implies that the price and delta of a European put are

$$P(S,t) = K\,e^{-r(T-t)}\,\mathrm{N}(-d_-) - S\,e^{-y(T-t)}\,\mathrm{N}(-d_+), \quad \Delta_p(S,t) = -e^{-y(T-t)}\,\mathrm{N}(-d_+)$$

and the analysis above shows that

$$\lim_{t\to T^-} P(S,t) = (K-S)^+, \quad \lim_{t\to T^-} \Delta_p(S,t) = \begin{cases} -1 & \text{if } 0<S<K, \\ 0 & \text{if } 0<K<S. \end{cases}$$

The risk-neutral price process

More generally the price a European option with payoff $P_o(S_T)$ is given by

$$V(S,t) = e^{-r(T-t)}\,\mathbb{E}_t\big[\,P_o(S_T)\,|\,S_t = S\,\big]$$

where S_t evolves according to (1). The process (1) is called *risk neutral* for the following reason.

First note that the $-y$ represents a continuous dividend yield. If this dividend is reinvested in the asset then the total value of the shares is $e^{yt}\,S_t$. Itô's lemma shows that

$$\begin{aligned} d(e^{yt}\,S_t) &= y\,e^{yt}\,S_t\,dt + e^{yt}\,dS_t \\ &= e^{yt}\,S_t\,\big(\,y\,dt + (r-y)\,dt + \sigma\,dW_t\,\big) \end{aligned}$$

so if we write the total value as $p_t = e^{yt}\,S_t$ then

$$\frac{dp_t}{p_t} = r\,dt + \sigma\,dW_t.$$

Now put $q_t = e^{-rt}p_t$ so Itô's lemma implies that

$$dq_t = -r\,e^{-rt}\,p_t\,dt + e^{-rt}\,dp_t = \sigma\,e^{-rt}\,p_t\,dW_t = \sigma\,q_t\,dW_t.$$

Integrating from s to $t \geq s$ we have

$$q_t - q_s = \sigma \int_s^t q_u\,dW_u$$

and taking expectations (with the information available at time s) gives

$$\mathbb{E}_s[\,q_t\,] - q_s = \mathbb{E}_s\Big[\int_s^t q_u\,dW_u\Big] = 0.$$

This implies that $\mathbb{E}_s\big[\,e^{-rt}\,p_t\,\big] = e^{-rs}\,p_s$ or

$$\mathbb{E}_s[\,p_t\,] = e^{r(t-s)}\,p_s,$$

that is, the total value of the (risky) portfolio of shares grows at the *risk-free* rate r under the process (1). We only use (1) for the purposes of pricing an option; it comes about because the Black-Scholes equation arises by eliminating all risk from the option (either by Δ-hedging it or by a self-financing replication strategy).

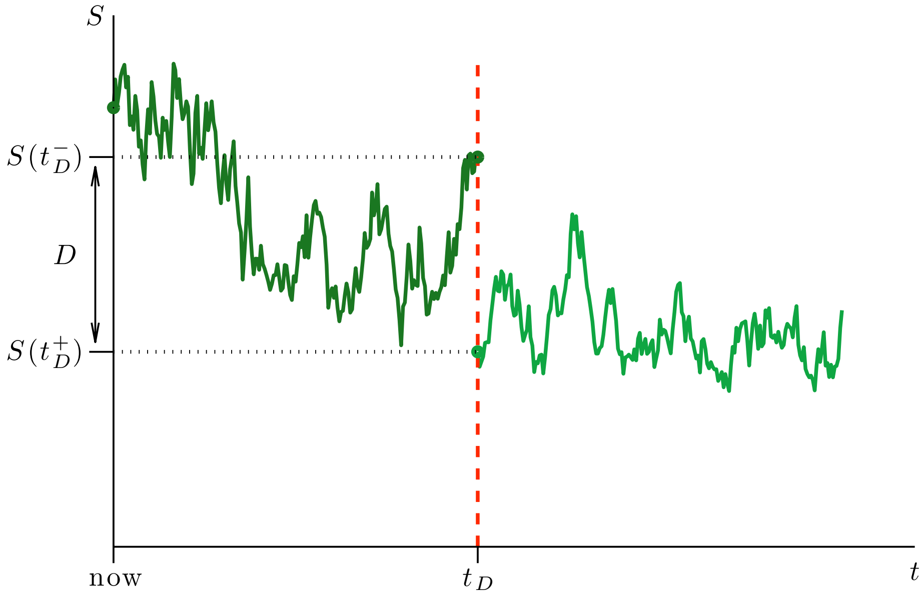

Figure 1: Jump in share price across a discrete dividend date.

Discrete dividends

Suppose that a share pays a deterministic dividend D at time t_D. If both D and t_D are known in advance we must have

$$S_{t_D^-} = S_{t_D^+} + D \quad \Longleftrightarrow \quad S_{t_D^+} = S_{t_D^-} - D$$

otherwise there is an arbitrage opportunity. If we have an option on this share then we *don't* get the dividend and so we must have the jump condition

$$V\big(S_{t_D^-}, t_D^-\big) = V\big(S_{t_D^+}, t_D^+\big) = V\big(S_{t_D^-} - D, t_D^+\big).$$

As this is true for any $S_{t_d^-}$ and we solve the Black-Scholes equation backwards in time, we generally write this *jump condition* as

$$V(S, t_D^-) = V(S - D, t_{D^+}). \tag{2}$$

The strategy is to solve the Black-Scholes equation back from expiry, T, until the dividend date t_D^+, then apply (2) to find $V(S, t_D^-)$ and then solve the Black-Scholes equation backwards from t_D^- to the present time, using $V(S, t_D^-)$ as a "payoff" at t_D^-.

Note that D can be a function of S and t. Indeed, if we want the share price to remain positive, it must be. Modelling discrete dividend payments for a share price that follows geometric Brownian motion is problematic to this day.

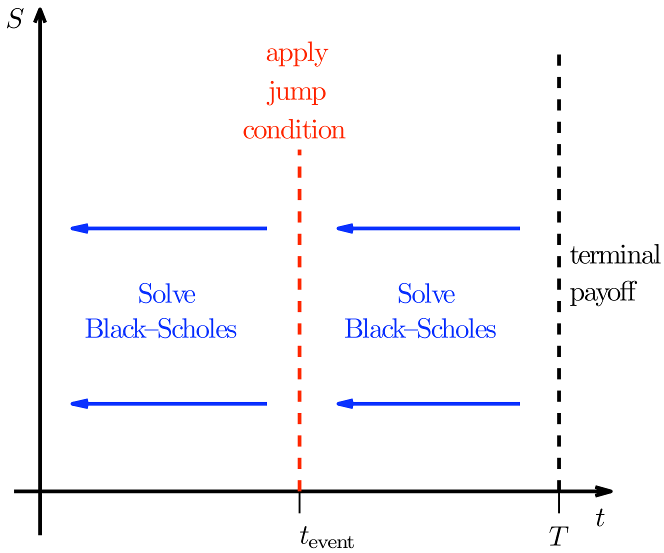

Figure 2: General strategy for dealing with a discrete-time event.

Discrete dividend yields

If we assume a discrete dividend of the form

$$D = d_y \, S_{t_d^-},$$

where the discrete dividend yield $d_y < 1$, i.e., the dividend is proportional to the share price immediately before the dividend is paid then we find that

$$S_{t_d^-} = S_{t_d^+} + d_y \, S_{t_d^-} \quad \Longleftrightarrow \quad S_{t_d^+} = (1 - d_y) \, S_{t_d^-}$$

and the jump condition for the option becomes

$$V(S, t_d^-) = V\big((1 - d_y)S, t_d^+\big). \tag{3}$$

We can then use the fact that if $V(S,t)$ is a solution of the Black-Scholes equation then so too is $V(\lambda S, t)$, with $\lambda = (1 - d_y)$ in this case, to see that the solution for $t < t_d$ is simply

$$V\big((1 - d_y)S, t\big),$$

as it is a solution of the Black-Scholes equation and obviously satisfies the "payoff" condition at t_d^-.

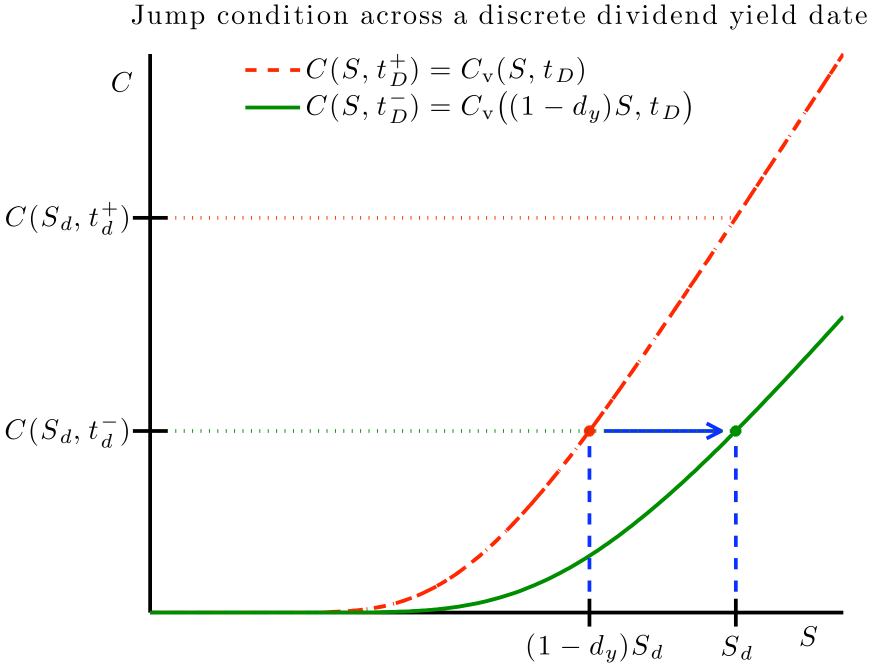

Figure 3: Jump condition for a call option on a share that pays a discrete dividend yield.

A call option with one discrete dividend yield

Let $C_{\mathrm{v}}(S,t)$ be the price function for a vanilla call, i.e.,

$$\begin{aligned} C_{\mathrm{v}}(S,t) &= S\,\mathrm{N}(d_+) - K\,e^{-r(T-t)}\,\mathrm{N}(d_-), \\ d_\pm &= \frac{\log(S/K) + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sqrt{\sigma^2(T-t)}.} \end{aligned}$$

Let the share pay a discrete dividend yield of d_y at time $0 < t_d < T$ and let $C(S,t)$ be the price function for a call written on this share. Then for $t_d < t < T$ we have

$$C(S,t) = C_{\mathrm{v}}(S,t).$$

Across the dividend date t_d, we apply (3) to get

$$C(S,t_d^-) = C_{\mathrm{v}}\big((1-d_y)S, t_d\big)$$

and then note that as $1-d_y > 0$ is a constant, the function $C_{\mathrm{v}}\big((1-d_y)S, t_d\big)$ is itself a solution of the Black-Scholes equation and so for all $t < t_d$ we have

$$C(S,t) = C_{\mathrm{v}}\big((1-d_y)S, t\big).$$

The same reasoning shows that if there are n discrete dividend yields at times

$$t < t_1 < t_2 < \cdots < t_n < T$$

between now and expiry with dividend yields

$$d_1,\ d_2,\ \ldots\ ,\ d_n,$$

where each $d_k < 1$, then

$$C(S,t) = C\big(\alpha_n S, t\big), \quad \text{where} \quad \alpha_n = \prod_{k=1}^{n} (1 - d_k).$$

Clearly this result generalises to any European option, regardless of the its payoff.