B8.3 Week 7 summary 2019

American options

An American option is an option which can be exercised at any time between being initiated and expiring (inclusive). It follows that

- It can not be less valuable than the payoff $P_{\mathrm{o}}(S_t, t)$, which may depend on t because the option can be exercised at any time $0 \le t \le T$. If it were, the arbitrage is to buy the option and immediately exercise it to receive the payoff (which is greater than the price).

- It can't be worth less than an otherwise equivalent European option. If it were, the arbitrage is to buy the American option, write the European option and put the positive profit in the bank. Then hold the American option until expiry (this may not be optimal, but if you own the American option you are free to do it) at which point it equals the European option and so you are perfectly covered.

It is easy to see that if $r > 0$ then for a European put we have

$$\lim_{S \to 0} P(S, t) = K\, e^{-r(T-t)} < K.$$

Since the European put price is differentiable, it is also continuous and so this shows that prior to expiry a European put is less valuable than the payoff for small enough S. As an American put can't be less valuable than the payoff, the values of American and European puts must be different. As they both have the same payoff, $(K - S)^+$, the American put can't satisfy the Black-Scholes equation for all $S > 0$.

Linear complementarity formulation (with $y = 0$)

There are a number of ways of formulating the American option problem. One is the *linear complementarity formulation,* which we give here. Let $V(S, t)$ be the value (function) of the option and $P_0(S, t)$ be the payoff (function). No arbitrage implies that

$$V(S, t) \ge P_{\mathrm{o}}(S, t), \quad S > 0,\ t \le T.$$

Go back to the derivation of the Black-Scholes pricing equation so at any time we hold one long position in the American option, V, and Δ_t short positions in the underlying asset. As for the European option, the market value is

$$M_t = V(S_t, t) - \Delta_t\, S_t$$

and the change in the value of the portfolio is

$$d\Pi_t = dV_t - \Delta_t\, dS_t.$$

Using Itô's lemma we get

$$d\Pi = \left(\frac{\partial V}{\partial t}(S_t, t) + \tfrac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2}(S_t, t) \right) dt + \left(\frac{\partial V}{\partial S}(S_t, t) - \Delta_t \right) dS_t$$

and so taking $\Delta_t = (\partial V/\partial S)(S_t, t)$ makes the change in portfolio value (instantaneously) risk-free. For the European option we then argued that $d\Pi_t > r\, M_t\, dt$ and $d\Pi_t < r\, M_t\, dt$ both represented arbitrage opportunities and hence $d\Pi_t = r\, M_t\, dt$, which gives the Black-Scholes equation.

For the American option it is still true that $d\Pi_t > r\, M_t\, dt$ gives a clear arbitrage; borrow the price of the portfolio, M_t, set up the portfolio with the correct value of Δ_t. At time $t + dt$ the portfolio's risk-free value is $M_t + d\Pi_t$ which is greater than $(1 + r\, dt)\, M_t$ than you owe. Therefore we must have $d\Pi_t \leq r\, M_t\, dt$ which is equivalent to the partial differential *inequality*

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r\, S\, \frac{\partial V}{\partial S} - r\, V \leq 0.$$

The problem comes with showing that $d\Pi_t < r\, M_t\, dt$ is an arbitrage if the option is American. This is because it involves short-selling or writing the option and, unlike a European option, an American option can be exercised at *any* time, not just at expiry. (Indeed, the only reason for exercising an American option before expiry is that the return on the delta-hedged portfolio is less than the return on the bank account.)

Now suppose that $V(S_t, t) > P_{\mathrm{o}}(S_t, t)$. Then it would be absurd to exercise the American option early as you could sell it for more. You could also short-sell it knowing that it wouldn't be exercised immediately. Therefore you *can* make an arbitrage if $d\Pi_t < r\, M_t\, dt$ and $V(S_t, t) > P_{\mathrm{o}}(S_t, t)$ and so

$$V(S, t) > P_o(S, t) \implies \mathcal{L}_{\mathrm{bs}}(V) = 0,$$

where $\mathcal{L}_{\mathrm{bs}}(V)$ is the Black-Scholes operator

$$\mathcal{L}_{\mathrm{bs}}(V) = \frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r\, S\, \frac{\partial V}{\partial S} - r\, V.$$

Now if $\mathcal{L}_{\mathrm{bs}}(V) < 0$ we can't have $V > P_{\mathrm{o}}$ for the reason immediately above, we can't have $V < P_{\mathrm{o}}$ as this represents an arbitrage and so the only possibility is $V(S, t) = P_{\mathrm{o}}(S, t)$. Thus

$$\mathcal{L}_{\mathrm{bs}}(V) < 0 \implies V(S, t) = P_{\mathrm{o}}(S, t).$$

In total we can write this as the *linear complementarity problem*

$$\mathcal{L}_{\mathrm{bs}}(V) \leq 0, \quad V(S, t) \geq P_{\mathrm{o}}(S, t),$$
$$\big(V(S, t) - P_{\mathrm{o}}(S, t) \big)\, \mathcal{L}_{\mathrm{bs}}(V) = 0.$$

At expiry we have $V(S, T) = P_{\mathrm{o}}(S, T)$. No arbitrage implies that $V(S, t)$ is continuous in S for $t < T$. We need another condition to uniquely determine $V(S, t)$ and it is that the *holder chooses the early exercise strategy in order to maximize the option's value.*

Smooth pasting

As shown in Merton (§8.7, pp 294–298) the fact that the holder chooses the early exercise strategy to maximize the option's value is *often* equivalent to the *smooth pasting conditions*, there is some $\hat{S}(t)$ such that

$$V\big(\hat{S}(t),t\big) = P_{\mathrm{o}}\big(\hat{S}(t),t\big), \quad \frac{\partial V}{\partial S}\big(\hat{S}(t),t\big) = \frac{\partial P_{\mathrm{o}}}{\partial S}\big(\hat{S}(t),t\big)$$

and on one side of $\hat{S}(t)$ we have $\mathcal{L}_{\mathrm{bs}}(V) = 0$ and on the other we have $V(S,t) = P_{\mathrm{o}}(S,t)$. The function $\hat{S}(t)$ is called the *optimal exercise* boundary. It is part of the problem to find $\hat{S}(t)$, hence the two conditions applied there.

Smooth pasting is *not* universally true. There are American options for which it is always true, there are some American options for which it is always false and there are other American options where it is sometimes true and sometimes false. For American puts and calls (calls with positive dividends $y > 0$) it is always true.

Smooth pasting for an American put

Consider an American put option (for simplicity, with no dividends) where the share price at time t is equal to the optimal exercise price, S_t^*. If $dS_t < 0$, so the share price goes down, then the put's value equals the payoff (and the option is exercised). If $dS_t > 0$, so the share price goes up, then the put's value is above the payoff (and the option is held). Thus we have

$$P(S_t^* + dS_t, t + dt) = \begin{cases} K - S^* - dS_t & \text{if } dS_t < 0, \\ P(S^* + dS_t, t + dt) & \text{if } dS_t > 0. \end{cases}$$

Now assuming that S_t follows a geometric Brownian motion,

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW_t,$$

where $dW_t \sim N(0, dt)$, it follows that $dW_t = \sqrt{dt}\, Z$ where $Z \sim N(0,1)$. Thus $dW_t = \mathcal{O}(\sqrt{dt})$ and since dt is infinitesimally small we have $dW_t \gg dt$. This in turn implies that

$$dS_t = \sigma\, S_t^*\, dW_t + \mathcal{O}(dt)$$

(recall that $S_t = S_t^*$) and

$$P(S_t^* + dS_t, t + dt) = P(S^*, t) + \sigma\, S_t^*\, \frac{\partial P}{\partial S}(S_t^*, t)\, dW_t + \mathcal{O}(dt).$$

Thus, with $P_t = P(S_t^*, t)$, we have

$$dP_t^* = \begin{cases} -\sigma\, S_t^*\, dW_t & \text{if } dW_t < 0, \\ \sigma\, S_t^*\, \dfrac{\partial P}{\partial S}(S^*, t)\, dW_t & \text{if } dW_t > 0. \end{cases}$$

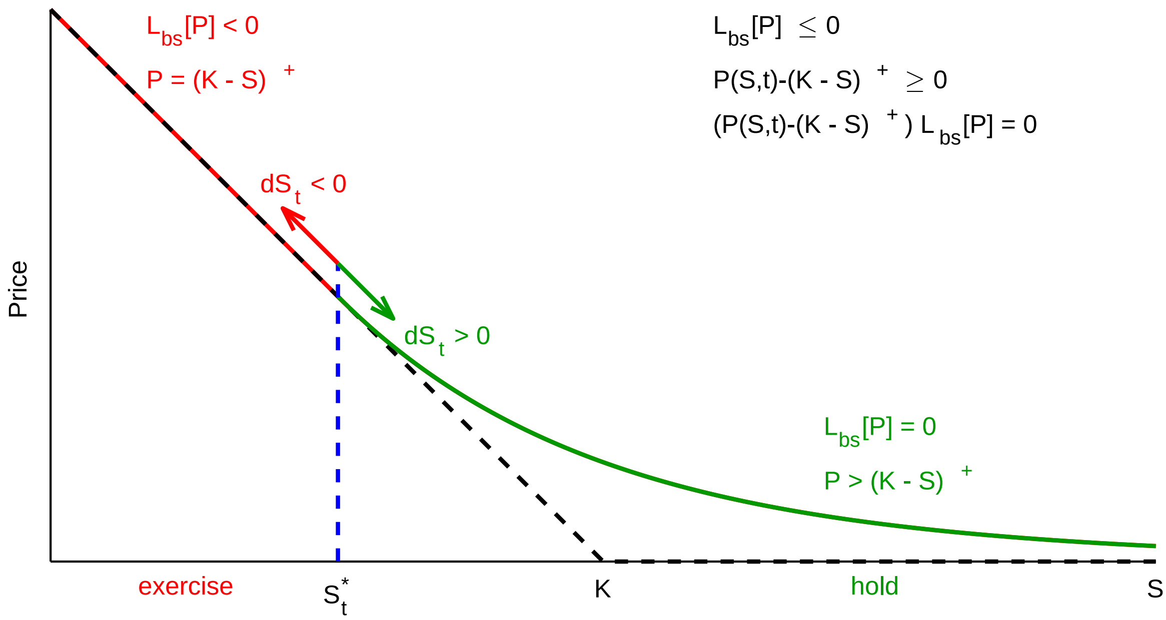

Figure 1: An American put option where the share price is equal to the optimal exercise price, S_t^*.

Consider a portfolio with a long put and a long share, $\Pi_t = P(S_t^*, t) + S_t^*$, (recall again that $S_t = S_t^*$). From the above we see that

$$d\Pi_t = \begin{cases} 0 & \text{if } dW_t < 0, \\ \sigma\, S_t^* \left(\dfrac{\partial P}{\partial S}(S^*, t) + 1 \right) & \text{if } dW_t > 0. \end{cases}$$

Now suppose that

$$\frac{\partial P}{\partial S}(S_t^*, t) + 1 > 0 \quad \text{or} \quad \frac{\partial P}{\partial S}(S_t^*, t) + 1 < 0.$$

Both of these cases lead to an arbitrage in which $d\Pi_t$ is either non-negative with a non-zero probability of being strictly positive (the first case) or non-positive with a non-zero probability of being strictly negative (the second case). Therefore, to avoid an arbitrage we must have

$$\frac{\partial P}{\partial S}(S_t^*, t) = -1,$$

which is the (second) smooth pasting condition.

Perpetual American options

We consider the case where the option never expires, $T \to \infty$. In this case there is no difference in the option pricing problem between the spot/time points (S, t_1) and (S, t_2) when $t_1 \neq t_2$, so, we can assume that $V = V(S)$. In this case, provided the option hasn't already been exercised, it satisfies the ordinary differential equation

$$\tfrac{1}{2}\sigma^2 S^2 V''(S) + (r - y) S V'(S) - r V(S) = 0.$$

This equation is sometimes called an Euler equation. One way to solve this equation is to look for solutions in terms of the eigenfunctions of $S \, \partial/\partial S$,

$$V(S) = S^m, \quad S V'(S) = m S^m, \quad S^2 V''(S) = m(m-1) S^m.$$

With this choice, we see that m must satisfy the quadratic equation

$$\tfrac{1}{2}\sigma^2 m(m-1) + (r - y) m - r = 0.$$

If we assume $\sigma > 0$, $r > 0$ and $y \geq 0$ and set

$$p(m) = \tfrac{1}{2}\sigma^2 m(m-1) + (r - y) m - r$$

we see that $p(m)$ is quadratic in m. Moreover, it has a positive coefficient for the quadratic term m^2 and at the points $m = 0$ and $m = 1$ we have $p(0) = -r < 0$ and $p(1) = -y \leq 0$. From these facts it follows that if $m^{\pm}$ are the roots of the quadratic then

$$m^- < 0, \quad m^+ \geq 1.$$

Thus the general solution is

$$V(S) = A S^{m^-} + B S^{m^+}, \quad m^- < 0, \; m^+ \geq 1.$$

Perpetual American put with smooth pasting

With $y \geq 0$ and $r > 0$ we find that the problem for the American put is

$$\tfrac{1}{2}\sigma^2 S^2 P''(S) + (r - y) S P'(S) - r P(S) = 0, \quad S > \hat{S},$$
$$P(\hat{S}) = K - \hat{S}, \quad P'(\hat{S}) = -1, \quad P(\infty) = 0.$$

The general solution is $P(S) = A S^{m^-} + B S^{m^+}$, where as above $m^- < 0$ and $m^+ \geq 1$. The two conditions $P(\hat{S}) = K - \hat{S}$ and $P(\infty) = 0$ give

$$P(S) = (K - \hat{S}) \left(\frac{S}{\hat{S}} \right)^{m^-}.$$

The remaining boundary condition, $P'(\hat{S}) = -1$, then gives

$$0 < \hat{S} = \frac{m^- K}{m^- - 1} < K$$

since $m^- < 0$ (which implies $m^- - 1 < m^- < 0$).

Perpetual American put by maximising

Again assume that $y \geq 0$ and $r > 0$. Choose an arbitrary $0 < \bar{S} < K$ and exercise as soon as S falls to $\bar{S}$. Then

$$\tfrac{1}{2}\sigma^2 S^2 P''(S) + (r-y)\, S\, P'(S) - r\, P(S) = 0, \quad S > \bar{S},$$
$$P(\bar{S}) = K - \bar{S}, \quad P(\infty) = 0.$$

As above, the solution is

$$P(S;\bar{S}) = (K-\bar{S})\left(\frac{S}{\bar{S}}\right)^{m^-},$$

where $m^- < 0$. Now (formally) set

$$\frac{\partial P}{\partial \bar{S}}(S;\bar{S}) = \left(\frac{S}{\bar{S}}\right)^{m^-}\left(-1 - m^-\,\frac{K-\bar{S}}{\bar{S}}\right) = 0$$

which gives

$$-1 - m^-\left(\frac{K-\bar{S}}{\bar{S}}\right) = 0$$

which in turn implies the optimal value of $\bar{S}$, $\hat{S}$, is

$$0 < \hat{S} = \frac{m^-\,K}{m^- - 1} < K.$$

This is the same as the smooth pasting version.

Perpetual American digital put

Assume $r > 0$ and $y \geq 0$. The problem for the perpetual American digital put option is

$$\tfrac{1}{2}\sigma^2 S^2 P_d''(S) + (r-y)\, S\, P_d'(S) - r\, P_d(S) = 0, \quad 0 < K = \hat{S} < S,$$
$$P_d(K) = 1, \quad P_d(\infty) = 0.$$

The solution is

$$P_d(S) = \left(\frac{S}{K}\right)^{m^-}.$$

It is *not* possible to make $P_d'(K)$ continuous at $S = K$. Thus, the second smooth pasting condition (involving $P_d'(\hat{S})$) *does not* apply in this case!

[Note that you *can not* adapt the smooth pasting argument used for the American put option, above, so that it works for an American digital put (perpetual or with finite expiry date T).]