B8.3 Week 8 summary 2019

Some exotic options

Exotic option is a catch-all term that includes options which are not generally traded on markets, or not widely traded. They often occur embedded in other more complex financial products, as structured products or as over-the-counter options created for clients with special financial needs.

Forward start calls

These involve a payoff with a strike, e.g., a call, where the strike is determined by the share price at some time T_1, where $0 < T_1 < T_2$, i.e., $K = S_{T_1}$. The trick to pricing them is to note that for $T_1 < t \le T_2$ the strike K is known. For a call, this means that for $T_1 < t \le T_2$ we can write

$$C_{\mathrm{fs}}(S,t) = S\, e^{-y(T_2-t)}\, \mathrm{N}(d_+) - K\, e^{-r(T_2-t)}\, \mathrm{N}(d_-), \quad K = S_{T_1}.$$

At time T_1, $K = S$ by definition and so

$$C_{\mathrm{fs}}(S,T_1) = A(r,y,\sigma,T_1,T_2)\, S$$

where $A(r,y,\sigma,T_1,T_2)$ is independent of S and t and is given by

$$A(r,y,\sigma,T_1,T_2) = e^{-y(T_2-T_1)}\, \mathrm{N}(d_+^*) - e^{-r(T_2-T_1)}\, \mathrm{N}(d_-^*),$$

$$d_\pm^* = \frac{(r - y \pm \frac{1}{2}\sigma^2)(T_2 - T_1)}{\sqrt{\sigma^2(T_2-T_1)}}.$$

Solving the Black-Scholes equation back from T_1 shows that for $t \le T_1$

$$C_{\mathrm{fs}}(S,t) = A(r,y,\sigma,T_1,T_2)\, S\, e^{-y(T_1-t)}.$$

Down and out barrier call options

A down and out barrier call option becomes worthless (colloquially refered to as "knocking out") if the share price falls to or below a barrier level, $B > 0$, at any time during the option's life. In these notes, we take B to be a constant. If $S_t > B$ for all $t \in [0,T]$ then it has payoff $(S_T - K)^+$. The pricing problem, assuming that the option has not already knocked out, is

$$\frac{\partial C_{\mathrm{do}}}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 C_{\mathrm{do}}}{\partial S^2} + (r-y)\, S\, \frac{\partial C_{\mathrm{do}}}{\partial S} - r\, C_{\mathrm{do}} = 0, \quad S > B,\ t < T,$$
$$C_{\mathrm{do}}(S,T) = (S-K)^+,\ S > B, \quad C_{\mathrm{do}}(B,t) = 0,\ t \le T.$$

The case $0 < B \le K$

The trick here is to recall that if $V(S,t)$ is a solution of the Black-Scholes equation then so too is

$$\hat{V}(S,t) = (S/B)^{2\alpha}\, V\big(B^2/S,t\big),$$

where $2\alpha = 1 - 2(r-y)/\sigma^2$, and that

$$\hat{V}(B,t) = (B/B)^{2\alpha}\, V\big(B^2/B,t\big) = V\big(B,t\big).$$

So if $C_{\mathrm{bs}}(S,t)$ is the price of a vanilla call option we see that

$$C_{\mathrm{do}}(S,t;B) = C_{\mathrm{bs}}(S,t) - (S/B)^{2\alpha} C_{\mathrm{bs}}\big(B^2/S,t\big)$$

is a solution of the Black-Scholes equation which satisfies $C_{\mathrm{do}}(B,t;B) = 0$. Then we notice that as $B < S$ and $B \le K$ we have $B^2/S < B \le K$ so that

$$C_{\mathrm{bs}}\big(B^2/S,T\big) = \big(B^2/S - K\big)^+ = 0$$

which shows that for $S > B$

$$C_{\mathrm{do}}(S,T;B) = C_{\mathrm{bs}}(S,T) = (S-K)^+.$$

Thus $C_{\mathrm{do}}(S,t;B)$ satisfies the pricing problem and is the Black-Scholes value of the barrier option.

The case $B > K > 0$

In this case the trick above fails because we find that $C(B^2/S,T) \neq 0$ for all $S > B$. The way to deal with it is to truncate the payoff of the call so that it becomes equal to zero if $S \le B$ but remains unchanged if $S > B$, i.e., replacing the vanilla call above with an option whose payoff is

$$V(S,T) = \begin{cases} 0 & \text{if } 0 < S \le B, \\ S - K & \text{if } S > B > K. \end{cases}$$

This payoff can be achieved by using a vanilla call with strike B plus $(B-K)$ digital calls, also with strike B. So instead of using the vanilla call price as above, we work with

$$V(S,t;B) = C_{\mathrm{bs}}(S,t;K=B) + (B-K)\, C_{\mathrm{d}}(S,t;K=B).$$

The Black-Scholes price function is given by

$$C_{\mathrm{do}}(S,t;B) = V(S,t;B) - (S/B)^{2\alpha} V\big(B^2/S,t\big)$$

since this satisfies the Black-Scholes equation and boundary condition at $S = B$ and if $S > B$ then $B^2/S < B$ and so $V\big(B^2/S,T\big) = 0$, giving us the correct payoff at T.

The down-and-in barrier call

This option remains worthless if the share price does *not* fall *below* the barrier $B > 0$ during the life of the option. If at some point during the life of the option we have $S_t < B$ then the option turns into a vanilla call with payoff $(S_T - K)^+$; this is often referred to as "knocking in". If we hold both a down-and-out and a down-and-in call option then we are guaranteed the payoff $(S_T - K)^+$ and so there is a down-and-in / down-and-out parity relation,

$$C_{\mathrm{do}}(S,t;B) + C_{\mathrm{di}}(S,t;B) = C_{\mathrm{bs}}(S,t)$$

and hence

$$C_{\mathrm{di}}(S,t;B) = C_{\mathrm{bs}}(S,t) - C_{\mathrm{do}}(S,t;B).$$

In the case that $B < K$ this simplifies to

$$C_{\mathrm{di}}(S,t;B) = (S/B)^{2\alpha}\, C_{\mathrm{bs}}\big(B^2/S,t\big).$$

Note that these formula for C_{di} are only valid if $S \geq B$ for all time up to the present. As soon as $S < B$ the option turns into a vanilla call and remains so until expiry.

Asian options

Asian options are options which depend on the average share price over the life of the option. In practice, it is usually the arithmetic average which we can define using the running sum of the share price

$$R_t = \int_0^t S_u\,du, \quad A_T = R_T/T = \frac{1}{T}\int_0^T S_u\,du,$$

where A_T is the average price at T. The option's price is a function of S_t, R_t and t, $V_t = V(S_t, R_t, t)$ for some function $V(S, R, t)$. If we note that

$$dR_t = S_t\,dt$$

and assume that $dW_t^2 = dt$ (which really means $d[W]_t = dt$) and perform a formal Taylor series expansion then we find

$$\begin{aligned} dV_t &= \Big(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2}\Big)\,dt + \frac{\partial V}{\partial R}\,dR_t + \frac{\partial V}{\partial S}\,dS_t \\ &= \Big(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} + S_t\,\frac{\partial V}{\partial R}\Big)\,dt + \frac{\partial V}{\partial S}\,dS_t, \end{aligned}$$

where all partial derivatives are evaluated at (S_t, R_t, t). Applying the usual hedging (or self-financing replication) argument(s) shows that to eliminate risk we must hold

$$\Delta_t = \Delta(S_t, R_t, t) = \frac{\partial V}{\partial S}(S_t, R_t, t)$$

shares at time t and that the pricing equation is

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-y)\, S \frac{\partial V}{\partial S} + S \frac{\partial V}{\partial R} - r\, V = 0. \tag{1}$$

This holds for all $S > 0$, $t < T$ and $R > 0$; as $S_u > 0$, $R_t = \int_0^t S_u \, du$ can take only positive values.

If the option is what is known as a floating-strike asian call, where the average plays the role of the strike, so the payoff is

$$V(S, R, T) = \big(S - R/T\big)^+,$$

then we can simplify the problem by working the variables

$$x = R/S, \quad V(S, R, t) = S\, u(x, t),$$

i.e., by pricing relative to the share price rather than a unit of currency (in finance this is usually called *a change of numeraire*). We find that

$$\frac{\partial V}{\partial t} = S \frac{\partial u}{\partial t}, \quad S \frac{\partial V}{\partial R} = S \frac{\partial u}{\partial x},$$
$$S \frac{\partial V}{\partial S} = S \left(u - x \frac{\partial u}{\partial x} \right), \quad S^2 \frac{\partial^2 V}{\partial S^2} = S\, x^2 \frac{\partial^2 u}{\partial x^2}.$$

Substituting these into the pricing equation (1) gives

$$\frac{\partial u}{\partial t} + \tfrac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} + \big(1 + (y-r)x\big) \frac{\partial u}{\partial x} - y\, u = 0. \tag{2}$$

At expiry, $t = T$, we have $V(S, R, T) = (S - R/T)^+ = S\, u(x, T)$ and so

$$u(x, T) = (1 - x/T)^+. \tag{3}$$

The Feynman-Kăc formula shows that the solution of problem (2)–(3) can be expressed as

$$u(x, t) = e^{-y(T-t)} \mathbb{E}\Big[\big(1 - x_T/T\big)^+ \,|\, x_t = x\Big],$$

where x_τ evolves as

$$dx_\tau = \big(1 + (y - r)\, x_\tau\big)\, d\tau + \sigma\, x_\tau \, dW_\tau,$$

for $\tau > t$, with $x_t = x$.

An exchange option

This is an option written on *two* shares with prices $S_{1,t}$ and $S_{2,t}$ and payoff

$$V(S_{1,T}, S_{2,T}, T) = \big(S_{1,T} - S_{2,T}\big)^+ = \max\big(S_{1,T} - S_{2,T}, 0\big).$$

In a Black-Scholes framework, we model the prices of the assets by

$$\frac{dS_{1,t}}{S_{1,t}} = (\mu_1 - y_1)\,dt + \sigma_1\,dW_{1,t}, \quad \frac{dS_{2,t}}{S_{2,t}} = (\mu_2 - y_2)\,dt + \sigma_2\,dW_{2,t},$$

where W_1 and W_2 are Brownian motions with covariation

$$[W_1, W_2]_t = \rho\,t, \quad d[W_1, W_2]_t = \rho\,dt, \quad \rho \in (-1, 1).$$

If $V_t = V(S_{1,t}, S_{2,t}, t)$ for some function $V(S_1, S_2, t)$, Itô's lemma is

$$\begin{aligned} dV_t &= \frac{\partial V}{\partial t}\,dt + \tfrac{1}{2}\frac{\partial^2 V}{\partial S_1^2}\,d[S_1]_t + \frac{\partial^2 V}{\partial S_1 \partial S_2}\,d[S_1, S_2]_t + \tfrac{1}{2}\frac{\partial^2 V}{\partial S_2^2}\,d[S_2]_t \\ &\qquad + \frac{\partial V}{\partial S_1}\,dS_{1,t} + \frac{\partial V}{\partial S_2}\,dS_{2,t} \\ &= \left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma_1^2\,S_{1,t}^2\,\frac{\partial^2 V}{\partial S_1^2} + \rho\,\sigma_1\,\sigma_2\,S_{1,t}\,S_{2,t}\,\frac{\partial^2 V}{\partial S_1 \partial S_2} + \tfrac{1}{2}\,\sigma_2^2\,S_{2,t}^2\,\frac{\partial^2 V}{\partial S_2^2}\right) dt \\ &\qquad + \frac{\partial V}{\partial S_1}\,dS_{1,t} + \frac{\partial V}{\partial S_2}\,dS_{2,t} \end{aligned}$$

We need to hedge the option with both shares, so we have two Δ_ts. Following a Black-Scholes type argument the market value of our portfolio is

$$M_t = V_t - \Delta_{1,t}\,S_{1,t} - \Delta_{2,t}\,S_{2,t}$$

and the cost of our hedging strategy evolves as

$$d\Pi_t = dV_t - \Delta_{1,t}\,dS_{1,t} - \Delta_{2,t}\,dS_{2,t} - y_1\,\Delta_{1,t}\,S_{1,t}\,dt - y_2\,\Delta_{2,t}\,S_{2,t}\,dt.$$

Setting

$$\Delta_{1,t} = \frac{\partial V}{\partial S_1}(S_{1,t}, t), \quad \Delta_{2,t} = \frac{\partial V}{\partial S_2}(S_{2,t}, t),$$

renders the hedged portfolio instantaneously risk-free and by the usual no-arbitrage argument this means we must have

$$d\Pi_t = r\,M_t\,dt.$$

After writing this out in full and noting that $S_{1,t}$ and $S_{2,t}$ can take any positive values, we arrive at the pricing equation

$$\begin{aligned} \frac{\partial V}{\partial t} \;&+\; \tfrac{1}{2}\sigma_1^2\,S_1^2\,\frac{\partial^2 V}{\partial S_1^2} \;+\; \rho\sigma_1\sigma_2\,S_1\,S_2\,\frac{\partial^2 V}{\partial S_1 \partial S_2} \;+\; \tfrac{1}{2}\sigma_2^2\,S_2^2\,\frac{\partial^2 V}{\partial S_2^2} \\ &+\; (r - y_1)\,S_1\,\frac{\partial V}{\partial S_1} \;+\; (r - y_2)\,S_2\,\frac{\partial V}{\partial S_2} \;-\; r\,V \;=\; 0, \end{aligned} \tag{4}$$

for $S_1 > 0$, $S_2 > 0$ and $t < T$. The payoff condition is

$$V(S_1, S_2, T) = (S_1 - S_2)^+.$$

In this particular case we can simplify the problem by pricing relative to one of the share prices, rather than relative to a unit of currency. If we write

$$x = S_1/S_2, \quad V(S_1, S_2, t) = S_2\, u(x, t)$$

we find that the payoff condition becomes

$$u(x, T) = (x - 1)^+$$

and the chain rule shows that $u(x, t)$ satisfies

$$\frac{\partial u}{\partial t} + \tfrac{1}{2}\hat{\sigma}^2\, x^2\, \frac{\partial^2 u}{\partial x^2} + (y_2 - y_1)\, x\, \frac{\partial u}{\partial x} - y_2\, u = 0,$$

where

$$\hat{\sigma}^2 = \sigma_1^2 - 2\rho\, \sigma_1\, \sigma_2 + \sigma_2^2 > 0.$$

This is the Black-Scholes equation with risk-free rate set to y_2, a continuous dividend yield of y_1 and a volatility of $\hat{\sigma}$. The payoff is that of a call with strike $K = 1$ and so

$$u(x, t) = x\, e^{-y_1(T-t)}\, \mathrm{N}(\hat{d}_+) - e^{-y_2(T-t)}\, \mathrm{N}(\hat{d}_-),$$

where

$$\hat{d}_\pm = \frac{\log(x) + (y_2 - y_1 \pm \frac{1}{2}\hat{\sigma}^2)(T-t)}{\sqrt{\hat{\sigma}^2(T-t)}}.$$

Unwinding the change of variables gives

$$V(S_1, S_2, t) = S_1\, e^{-y_1(T-t)}\, \mathrm{N}(\hat{d}_+) - S_2\, e^{-y_2(T-t)}\, \mathrm{N}(\hat{d}_-)$$

with

$$\hat{d}_\pm = \frac{\log(S_1/S_2) + (y_2 - y_1 \pm \frac{1}{2}\hat{\sigma}^2)(T-t)}{\sqrt{\hat{\sigma}^2(T-t)}}.$$