B8.2: Continuous Martingales and Stochastic Calculus (2019)
Problem Sheet 1

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The questions on this sheet are divided into two sections. Those in the first section are compulsory and should be handed in for marking. Those in the second are extra practice questions and should not *be handed in.*

The questions are not in order of difficulty; if you are stuck on one question, move on to the next.

Section 1 (Compulsory)

1. Let $(X_i : i \in I)$ be a collection of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{H} = \sigma(X_i : i \in I)$ and $\mathcal{G} \subset \mathcal{F}$ a σ-algebra. Use the Monotone Class Lemma (see Appendix in the lecture notes for a reminder) to argue that:

   (a) In order to verify that $\mathcal{G}$ and $\mathcal{H}$ are independent it is enough to verify that $(X_{i_1}, \dots, X_{i_k})$ is independent of $\mathcal{G}$ for any finite set of indices $\{i_1, \dots, i_k\} \subset I$.

   (b) In order to verify that for a bounded real random variable Y, and a fixed $i_0 \in I$, we have $\mathbb{E}[Y|\sigma(X_{i_0})] = \mathbb{E}[Y|\mathcal{H}]$, it is enough to show that $\mathbb{E}[Y|\sigma(X_{i_0})] = \mathbb{E}[Y|\sigma(X_{i_0}, X_{i_1}, \dots, X_{i_k})]$ for any finite set of indices $\{i_1, \dots, i_k\} \subset I$.

2. Let X and Y be jointly Gaussian random variables in $\mathbb{R}^d$ and $\mathbb{R}^k$ respectively, where $k, d \in \mathbb{N}$. Suppose $\mathbb{E}[X] = \mu_X$, $\mathbb{E}[Y] = \mu_Y$, $\mathrm{Var}(X) = \Gamma_X$, $\mathrm{Var}(Y) = \Gamma_Y$ and

$$\mathrm{cov}(X, Y) = \mathbb{E}[(X - \mu_X)(Y - \mu_Y)^\top] = \Gamma_{XY} \in \mathbb{R}^{d \times k}.$$

   Suppose Γ_X, Γ_Y and $\Gamma_Y - \Gamma_{XY}^\top \Gamma_X^{-1} \Gamma_{XY}$ are all strictly positive definite. By considering the joint density or otherwise, show that Y is conditionally Gaussian given X, and give its conditional mean and variance.

   Hint: The following matrix identity (from block-matrix inversion) may simplify calculations:

$$\begin{bmatrix} x \\ y \end{bmatrix}^\top \begin{bmatrix} \Gamma_X & \Gamma_{XY} \\ \Gamma_{XY}^\top & \Gamma_Y \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = x\Gamma_X^{-1}x + (y - \Gamma_{XY}^\top \Gamma_X^{-1} x)^\top (\Gamma_Y - \Gamma_{XY}^\top \Gamma_X^{-1} \Gamma_{XY})^{-1} (y - \Gamma_{XY}^\top \Gamma_X^{-1} x).$$

3. Suppose that under the probability measure $\mathbb{P}$, the discrete time process $(S_n)_{n \geq 0}$ is a simple symmetric random walk on $\mathbb{Z}$. Show that $(S_n)_{n \geq 0}$ is a $\mathbb{P}$-martingale (with respect to the natural filtration) and

$$\mathrm{cov}(S_n, S_m) = n \wedge m.$$

4. Suppose that $(B_t)_{t \geq 0}$ is a Brownian motion. Fix $0 \leq s < t < \infty$. Show that conditionally on $\{B_s = x, B_t = z\}$ the intermediate value $B_{\frac{t+s}{2}}$ has Gaussian distribution with mean $\frac{x+z}{2}$ and variance $\frac{t-s}{4}$.

5. Suppose that $(B_t)_{t\geq 0}$ is a Brownian motion. Show that $(-B_t : t \geq 0)$ is a Brownian motion. Show that for any $c > 0$, $(cB_{t/c^2} : t \geq 0)$ is also a Brownian motion.

6. Suppose that $(B_t)_{t\geq 0}$ is a Brownian motion. Define a process X by $X_0 = 0$ and $X_t := tB_{1/t}$, $t > 0$. Show that X is a centred Gaussian process and specify its covariance function.

Section 2 (Extra practice questions, not for hand-in)

A. Suppose that ξ is normally distributed with mean zero and variance one and that $x > 0$. Show that

$$\mathbb{P}[\xi \geq x] \leq \frac{1}{x\sqrt{2\pi}} e^{-x^2/2}.$$

B. Show, using the Strong Law of Large Numbers, that $\frac{1}{n} B_n \to 0$ a.s. when $n \to \infty$, $n \in \mathbb{N}$.

C. Using the monotone class theorem or otherwise, show that any set $A \subset \mathbb{R}^{[0,\infty)}$ in the Borel cylinder σ-algebra is of the form

$$A = \bigcup_{i \in \mathbb{N}} \bigcap_{j \in \mathbb{N}} \{X : X_{t_{i,j}} \in B_{i,j}\}$$

for some countable set of times $\{t_{i,j}\}_{i,j \in \mathbb{N}}$ and sets $\{B_{i,j} \in \mathcal{B}(\mathbb{R})\}_{i,j \in \mathbb{N}}$. Consequently, show that the set $\{X \in \mathbb{R}^{[0,\infty)} : X \text{ is continuous}\}$ is not in $\mathcal{B}(\mathbb{R}^{[0,\infty)})$

D. Let X be a Gaussian process, with expectation $\mu(t) = \mathbb{E}[X_t]$ and covariance $\text{cov}(X_s, X_t) = \Gamma(s, t)$. Suppose X has continuous paths. Let $Y_t = \int_0^t X_s ds$. Show that Y is a Gaussian process, with parameters

$$\mathbb{E}[Y_t] = \int_0^t \mu(s) ds, \qquad \text{cov}(Y_s, Y_t) = \int_0^s \int_0^t \Gamma(s', t') dt' \, ds'.$$

Hence conclude that, for any Gaussian process with continuously differentiable mean and covariance, there is a construction of the process with differentiable paths.

(The stronger statement that every realization of such a process has a differentiable modification is true but is more difficult to prove.)