B8.2: Continuous Martingales and Stochastic Calculus (2019)
Problem Sheet 3

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The questions on this sheet are divided into two sections. Those in the first section are compulsory and should be handed in for marking. Those in the second are extra practice questions and should not *be handed in.*

The questions are not in order of difficulty; if you are stuck on one question, move on to the next.

Section 1 (Compulsory)

1. Let $(x_n)_{n\geq 1}$ be a sequence of real numbers. For real numbers $a < b$, let $U([a,b],(x_n)_{n\geq 1})$ be the number of upcrossings of $[a,b]$ by the sequence. Show that $(x_n)_{n\geq 1}$ converges to a limit in $[-\infty,\infty]$ as $n\to\infty$ if and only if $U([a,b],(x_n)_{n\geq 1}) < \infty$ for all $a < b$ with $a,b\in\mathbb{Q}$.

2. Let H_a be the first hitting time of a, $H_a = \inf\{t\geq 0 : B_t = a\}$. (We retain this notation for the next two questions.)

   Use Optional Stopping to compute the distribution of $B_{H_a\wedge H_b}$ for $a < 0 < b$.

3. Recall that we used Optional Stopping to show that the Laplace transform of H_a is given by
$$\mathbb{E}\left[e^{-\lambda H_a}\right] = e^{-|a|\sqrt{2\lambda}} \quad a\in\mathbb{R}, \lambda > 0.$$
   Consider $a, b > 0$. Deduce that if ξ_a, ξ_b are independent and distributed as H_a and H_b respectively, then $\xi_a + \xi_b$ has the same distribution as H_{a+b}. Use the strong Markov property to find an alternative proof of this result.

4. Use Optional Stopping to show that the Laplace transform of $H_a \wedge H_{-a}$ is given by
$$\mathbb{E}\left[e^{-\lambda H_a\wedge H_{-a}}\right] = \frac{1}{\cosh(a\sqrt{2\lambda})}, \quad a > 0, \lambda > 0.$$
   Hint: use martingales $(M^{(\theta)} + M^{(-\theta)})/2$*, where* $M_t^{(\theta)} := \exp(\theta B_t - \theta^2 t/2)$.

5. Let M be a positive continuous martingale converging a.s. to zero as $t\to\infty$. Let $M^* := \sup_{t\geq 0} M_t$. Note that we do not assume that $\mathcal{F}_0$ is trivial or that M_0 is deterministic.

   (a) For $x > 0$, prove that
$$\mathbb{P}\left[M^*\geq x|\mathcal{F}_0\right] = 1\wedge\frac{M_0}{x}.$$
   Conclude that M^* has the same distribution as M_0/U, where U is independent of M_0 and uniformly distributed on $[0,1]$.
   (*Hint: stop* M *when it becomes larger than* x).

(b) Let $a > 0$ and $B_t^a := a + B_t$ be a Brownian motion started at a. Let $\tau = H_0(B^a) = H_{-a}(B) = \inf\{t \geq 0 : B_t^a = 0\}$. Find the distribution of the random variable $Y := \sup_{t \leq \tau} B_t^a$.

6. Suppose that $(B_t)_{t \geq 0}$ is Brownian motion under $\mathbb{P}$. For a partition π of $[0, T]$, write $\|\pi\|$ for the mesh of the partition and $0 = t_0 < t_1 < t_2 < \ldots < t_{N(\pi)} = T$ for the endpoints of the intervals of the partition. Calculate

   (a)
$$\lim_{\|\pi\| \to 0} \sum_{j=0}^{N(\pi)-1} B_{t_{j+1}} \left(B_{t_{j+1}} - B_{t_j} \right),$$

   (b)
$$\int_0^T B_s \circ dB_s \triangleq \lim_{\|\pi\| \to 0} \sum_{j=0}^{N(\pi)-1} \frac{1}{2} \left(B_{t_{j+1}} + B_{t_j} \right) \left(B_{t_{j+1}} - B_{t_j} \right).$$

   This is the *Stratonovich integral* of $\{B_s\}_{s \geq 0}$ with respect to itself over $[0, T]$.

Section 2 (Extra practice questions, not for hand-in)

A. Suppose that the real-valued function a is of bounded variation and that f is a-integrable. Show that the function $(f \cdot a)$ defined by
$$(f \cdot a)(t) = \int_0^t f(s) da(s)$$
is right continuous and of finite variation.

B. Let a be a function of finite variation, $a(0) = 0$ and $f : [0, T] \to \mathbb{R}$ a continuous function. Show that
$$\int_0^T f(s) da(s) = \lim_{n \to \infty} \sum_{i=0}^{m_n - 1} f(t_i^n)(a(t_{i+1}^n) - a(t_i^n)),$$
where $\pi_n = \{0 = t_0 < t_1 < \ldots < t_{m_n} = T\}$ is a sequence of partitions of $[0, T]$ with $mesh(\pi_n) \to 0$ as $n \to \infty$.
(*Hint: use dominated convergence theorem for the associated measures* μ_+ *and* μ_-, *where* $\mu([0, t]) = a(t)$ *and* $\mu = \mu - \mu_-$).

C. Recall that an adapted right-continuous stochastic process M is called a *local martingale* if there exists a sequence of stopping times $\tau_n \uparrow \infty$ a.s. such that for any n the stopped process M^{τ_n} is a martingale, where $M_t^{\tau_n} = M_{\tau_n \wedge t}$. The sequence (τ_n) is called a *reducing sequence* or a *localising sequence*. Show that if M is a local martingale then

   (a) if M is non-negative (i.e. $\forall t \geq 0$ $M_t \geq 0$ a.s.) then it is a supermartingale;

   (b) M is a (true) martingale if and only if for any $a > 0$ the family
$$\{M_\tau : \tau \text{ a stopping time with } \tau \leq a\}$$
   is uniformly integrable.

D. (**A primer in stochastic integration**: *We define here the stochastic integral of a simple process w.r.t. to a nice martingale. The problem may appear long but is elementary and its aim is primarily for you to reflect on desirable properties a* stochastic integral *should have.*)

Let $M = (M_t : t \geq 0)$ be a uniformly integrable martingale bounded in L^2: $\sup_{t\geq 0} \mathbb{E}[M_t^2] < \infty$. Let $\mathcal{E}$ be the space of simple bounded process of the form

$$\varphi_t = \sum_{i=0}^{m} \varphi^{(i)} \mathbf{1}_{(t_i, t_{i+1}]}(t), \quad t \geq 0$$

for some $n \in \mathbb{N}$, $0 \leq t_0 < t_1 < \ldots < t_{m+1}$ and where $\varphi^{(i)}$ are bounded $\mathcal{F}_{t_i}$ measurable random variables. Define the stochastic integral $\varphi \cdot M$ of such φ with respect to M via

$$(\varphi \cdot M)_t := \sum_{i=0}^{m} \varphi^{(i)}(M_{t\wedge t_{i+1}} - M_{t\wedge t_i}), \quad t \geq 0.$$

(a) Show that $\varphi \to \varphi \cdot M$ is linear on $\mathcal{E}$;

(b) Show that $\varphi \cdot M$ is a martingale for all $\varphi \in \mathcal{E}$;

(c) Let τ be a stopping time and recall that for a process X, X^τ is the stopped process $X_t^\tau = X_{\tau\wedge t}$. Show that we have equality between the following three processes:

$$\varphi \cdot M^\tau = (\varphi \cdot M)^\tau = (\mathbf{1}_{[0,\tau]}\varphi \cdot M),$$

and where to define the last integral we would extend the definition of $\mathcal{E}$ to the case of (t_i) being a sequence of bounded stopping times.

(d) Compute $\mathbb{E}[(\varphi \cdot M)_t^2]$ for a $\varphi \in \mathcal{E}$. Conclude that $\sup_{t\geq 0} \mathbb{E}[(\varphi \cdot M)_t^2] < \infty$.

Now assume that there is an adapted non-decreasing process $\langle M \rangle$, $\langle M \rangle_0 = 0$ a.s. and such that $(M_t^2 - \langle M \rangle_t : t \geq 0)$ is a martingale. Show that

$$\mathbb{E}[(\varphi \cdot M)_t^2] = \mathbb{E}\left[\int_0^t \varphi_s^2 d\langle M \rangle_s\right].$$

(e) Let $\varphi, \psi \in \mathcal{E}$. Show that $\psi\varphi \in \mathcal{E}$ and that $\psi \cdot (\varphi \cdot M) = (\psi\varphi \cdot M)$.