B7.3 Further Quantum Theory

Problem Sheet 1

Hilary Term 2019

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1.1 Harmonic oscillator (revision)

Consider a quantum mechanical particle moving in one spatial dimension and introduce the operators $a_\pm = P \pm im\omega X$. Show that the simple harmonic oscillator Hamiltonian

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2 \ ,$$

can be expressed as

$$H = \frac{1}{2m}a_+a_- + \frac{\hbar\omega}{2} = \frac{1}{2m}a_-a_+ - \frac{\hbar\omega}{2} \ .$$

Also show that

$$\|a_\pm\psi\|^2 = 2mE_\psi(H) \pm m\hbar\omega \ .$$

Finally show that

$$[a_-, a_+] = 2m\omega\hbar \ , \quad [H, a_\pm] = \pm\hbar\omega a_\pm \ .$$

Suppose that there is a state ψ_0 so that $a_-\psi_0 = 0$. Show that $\psi_n = (a_+)^n\psi_0$ is an energy eigenstate with $E_n = (n + \frac{1}{2})\hbar\omega$. Comment briefly on the existence and uniqueness of these energy eigenstates.

1.2 Angular momentum (revision)

Let L_i be the components of the angular momentum operator $\mathbf{L} = \mathbf{X} \wedge \mathbf{P}$. Show that $[L_1, L_2] = i\hbar L_3$ and cyclic. Define $L_\pm = L_1 \pm iL_2$ and show that

$$[L_3, L_\pm] = \pm\hbar L_\pm \ , \qquad [L_+, L_-] = 2\hbar L_3 \ .$$

Deduce that $L_\pm$ are raising and lowering operators for the eigenvalues of L_3. Prove the identity

$$L_+L_- = \mathbf{L}\cdot\mathbf{L} - L_3^2 + \hbar L_3 \ .$$

Show further that if both $\mathbf{L}^2|\psi\rangle = \lambda\hbar^2|\psi\rangle$ and $L_3|\psi\rangle = m\hbar|\psi\rangle$ then we have

$$\langle L_-\psi|L_-\psi\rangle = (\lambda - m^2 + m)\hbar^2.$$

What can you say about the possible values of m?

1.3 Potential scattering in one dimension

Calculate transmission and reflection coefficients for a beam of particles incident from the left ($x = -\infty$) upon a potential barrier

$$V(x) = \begin{cases} V_0 & \text{if} \quad x \in [0, a] \\ 0 & \text{if} \quad x \notin [0, a] \ . \end{cases}$$

Distinguish the cases when $E > V_0$ and $0 < E < V_0$. Define $k^2 = 2mE/\hbar^2$ and for the former case define $k_0^2 = 2m(E - V_0)/\hbar^2$, while for the latter define $\kappa_0^2 = 2m(V_0 - E)$. Determine the values of a for which the transmission coefficient is maximum or minimum.

1.4 WKB approximation and Bohr-Sommerfeld

Use the WKB approximation and the Bohr-Sommerfeld quantization condition to derive a formula for the energy levels of

(a) the harmonic oscillator, with total energy given by

$$E = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \ ,$$

(b) for the s-orbital bound states (*i.e.*, spherically symmetric wave functions) of a particle of mass m in a Coulomb potential, so with energy

$$E = \frac{p_r^2}{2m} - \frac{e^2}{4\pi\epsilon_0 r} \ .$$

You may wish to parametrize the classical orbits in these systems as:

(a) $(p, x) = \sqrt{2mE}\left(\sin\theta, \frac{1}{\omega m}\cos\theta\right)$

(b) $(p_r, r) = \left(\sqrt{-2mE}\cot\theta, -\frac{e^2}{4\pi\epsilon_0 E}\sin^2\theta\right)$.

1.5 Bohr-Sommerfeld and state-counting

Use the discussion following the Bohr-Sommerfeld quantization rule in the lectures/notes to estimate the number of states of energy less than $n\hbar\omega$ in the simple harmonic oscillator in one and two dimensions. How good are the estimates?

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