Further Quantum Theory: Problem Sheet 3

Hilary Term 2018

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3.1 Spin $\frac{1}{2}$

(i) The spin operator $\mathbf{S}$ is Hermitian and satisfies the commutation relations

$$[S_1, S_2] = i\hbar S_3 \ , \quad [S_2, S_3] = i\hbar S_1 \ , \quad [S_3, S_1] = i\hbar S_2 \ ,$$

and for a spin $\frac{1}{2}$ particle, one has $\mathbf{S}^2 = \frac{3}{4}\hbar^2$. Show that the eigenvalues of S_3 are $\pm\frac{1}{2}\hbar$.

(ii) Assuming that the eigenvalues of S_3 are non-degenerate, show that by suitable choice of phase

$$S_-\psi^{\frac{1}{2}} = \hbar\psi^{-\frac{1}{2}} \quad \text{and} \quad S_+\psi^{-\frac{1}{2}} = \hbar\psi^{\frac{1}{2}} \ ,$$

where $S_\pm = S_1 \pm iS_2$ and $\psi^{\pm\frac{1}{2}}$ are the eigenvectors of S_3 corresponding to the eigenvalues $\pm\frac{1}{2}\hbar$.

(iii) Two spin $\frac{1}{2}$ particles with spin operators $\mathbf{S}(1)$ and $\mathbf{S}(2)$ are coupled so that their Hamiltonian is

$$H = k\mathbf{S}(1) \cdot \mathbf{S}(2) \ .$$

Show that

$$\mathbf{S}(1) \cdot \mathbf{S}(2) = \frac{S_+(1)S_-(2) + S_-(1)S_+(2)}{2} + S_3(1)S_3(2) \ .$$

Hence or otherwise obtain the energy eigenvalues for this system.

(iv) What are the allowed eigenvalues and their degeneracies if the particles are identical and satisfy (a) Fermi or (b) Bose statistics?

3.2 Spin + orbit

Suppose that an electron is in a state of orbital angular momentum $l = 2$. Show how to construct the state vectors Ψ_j^m with total angular momentum $j = 5/2$ and corresponding 3-components of its angular momentum $m = 5/2$ and $m = 3/2$ as linear combinations of state vectors with definite values of S_3 and L_3. Do the same for Ψ_j^m with $j = 3/2$ and $m = 3/2$. (All state vectors here should be properly normalized.)

Summarize your results by providing the ClebschGordan coefficients $C_{\frac{1}{2},2}(j, m; m_s, m_l)$ when $(j, m) = (5/2, 5/2), (5/2, 3/2)$, and $(3/2, 3/2)$.

3.3 Spin-orbit coupling

The spin-orbit coupling of the electron in a hydrogen atom leads to a term in the Hamiltonian of the form

$$V_{SO} = \xi(r)\mathbf{L} \cdot \mathbf{S} \ ,$$

where $\xi(r)$ is some small function of r. Give an integral formula for the contribution of V_{SO} to the fine-structure splitting between the $2p_{1/2}$ and $2p_{3/2}$ states in hydrogen to first order in ξ. *You may use the fact that the radial part of the $2p$ wave function for the electron in the hydrogen atom has the form* $re^{-r/2a}/(2\sqrt{6}a^{5/2})$.

3.4 Time-independent perturbation theory

A correction term H' is added to the Hamiltonian H_0 of some quantum mechanical system with a small coefficient λ: $H_0 + \lambda H'$. Assume that the energy eigenvalues and eigenstates of the perturbed system can be expanded in a power series in λ,

$$\begin{aligned} E &= E_0 + \lambda E' + \lambda^2 E'' + \dots \ , \\ \psi &= \psi_0 + \lambda \psi' + \lambda^2 \psi'' + \dots \ , \end{aligned}$$

Find the first order correction to the nondegenerate energy level E_0. Show that when $\phi = H'\psi_0$ is an eigenfunction of H_0 with energy $\mathcal{E} \neq E_0$ the first order energy correction vanishes and the first order correction to the wave function is a multiple of ϕ which should be found. Assuming that ψ_0 is normalised, show also that to second order the energy is

$$E_0 + \frac{\lambda^2 \|\phi\|^2}{E_0 - \mathcal{E}} \ .$$

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