Further Quantum Theory: Problem Sheet 4

Hilary Term 2018

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4.1 Exactly perturbed harmonic oscillator

A particle of mass m moves in two dimensions under the influence of a potential $V(x,y) = \frac{1}{2}m\omega^2(x^2 + y^2 + 2\epsilon xy)$.

(*i*) What are the possible energy levels of the Hamiltonian H_0 when $\epsilon = 0$? What is the ground state wave function ψ_0?

You may assume that the two lowest states of the harmonic oscillator with Hamiltonian $P^2/2m + \frac{1}{2}m\omega^2 X^2$ *have normalised wave functions*

$$\phi_0(x) = (\pi a^2)^{-\frac{1}{4}} \exp(-x^2/2a^2), \qquad \phi_1(x) = (\sqrt{2}x/a)\phi_0(x) \ ,$$

with energies $\frac{1}{2}\hbar\omega$ *and* $\frac{3}{2}\hbar\omega$*, respectively, where* $a^2 = \hbar/m\omega$.

(*ii*) When ϵ does not vanish, and the Hamiltonian takes the form $H = H_0 + \epsilon H'$. Show that $H'\psi_0$ is an eigenvector of H_0. Using the previous question, or otherwise, find the second order correction to the energy of the ground state.

(*iii*) Show that the first excited state of the unperturbed system is degenerate and use degenerate perturbation theory to calculate the first order correction to the energy of the first excited state to order ϵ. Calculate the energy levels of the system directly and compare the exact answer with that given by perturbation theory.

4.2 Rayleigh-Ritz

The total angular momentum operator in spherical polar coordinates (and atomic units: $\hbar = 1$), has the form

$$L^2 = -\frac{1}{\sin^2\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{\partial^2}{\partial\phi^2}\right) \ .$$

Using 1 and $\cos^2\theta$ as the basis for a space of trial functions, obtain the Rayleigh-Ritz estimates for the eigenvalues of L^2. Show that the approximate eigenvectors obtained by this technique are in fact true eigenvectors.

Recall that the Rayleigh-Ritz method for estimating the eigenvalues of a self-adjoint operator A*, uses trial functions in the space spanned by* $\phi_1, \phi_2, \ldots, \phi_N$ *and leads to the secular equation* $\det(\langle\phi_j|A\phi_k\rangle - \lambda\langle\phi_j|\phi_k\rangle) = 0$.

4.3 Heisenberg-picture operators

The Hamiltonian for the one-dimensional harmonic oscillator is given by

$$H = \frac{1}{2m}\left(P^2 + m^2\omega^2X^2\right) \ ,$$

and as usual we can define the ladder operators $A^\pm = (P \pm im\omega X)$. Show that in the Heisenberg picture, these ladder operators evolve according to

$$A_t^\pm = e^{\pm i\omega t}A_0^\pm \ .$$

Show further that

$$\langle X^2\rangle_{t+\pi/2\omega} + \langle X^2\rangle_t = \text{constant} \ ,$$

where $\langle X^2\rangle_t$ is the expectation value of X^2 at time t.

4.4 Interaction picture

The Hamiltonian for a quantum system is given by

$$H = \frac{P^2}{2m} + \frac{1}{2} m\omega^2 X^2 + \epsilon X \ .$$

(i) Show that in the interaction picture with $H' = \epsilon X$ and $H_0 = H - H'$, the wave function at time t is given by

$$\psi_t = \psi_0 - \frac{i}{\hbar} \int_0^t H'_s \psi_s \, ds, \quad \text{where} \quad H'_s = i \frac{\epsilon}{2m\omega} \left(e^{-i\omega s} A_0^- - e^{i\omega s} A_0^+ \right) \ .$$

(You may use results from the previous question.)

(ii) Let ψ_0 be the ground state of the harmonic oscillator H_0. Show that to first order in ϵ the wave function at a later time is

$$\psi_t = \psi_0 + i \frac{\epsilon (e^{i\omega t} - 1)}{2m\hbar\omega^2} A_0^+ \psi_0 \ .$$

and that the probability of transition to the first excited state $(2m\hbar\omega)^{-\frac{1}{2}} A_0^+ \psi_0$ is approximately

$$\frac{2\epsilon^2}{m\hbar\omega^3} \sin^2 \frac{\omega t}{2} \ .$$

4.5 Time-dependent perturbation theory

Consider a time-*dependent* Hamiltonian $H = H_0 + H'(t)$, with $H'(t) = U \exp(-t/T)$, where H_0 and U are time-independent operators and T is a constant with units of time. What is the probability to lowest order in U that the perturbation will produce a transition from one eigenstate ϕ_n of H_0 to a different eigenstate ϕ_m of H_0 starting at $t = 0$ as $t \to \infty$?

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