Numerical Solution of Differential Equations II. QS 1

Question 1. By (artificially) employing analytic solution methods for initial value problems (but not for the BVP) solve

$$u'' - 2u' + 2u = 2, \quad u(0) = 1, \quad u(\frac{\pi}{2}) = 2$$

by the shooting method for linear problems. (You may find that selection of initial slopes $u_0'(0) = 0$ and $u_1'(0) = 1$ where u_0 solves the inhomogeneous IVP and u_1 solves the homogeneous IVP makes the calculation a little easier, although any values should work!)

Question 2. As Qn 1 above; why would there be a difficulty if the right hand boundary condition was set to $u(\pi) = 2$? What solution(s) exist in this case?

Question 3. Show that solving

$$u'' = 2u' - 2u + 2, \quad u(0) = 1, \quad u(\frac{\pi}{2}) = 2$$

by the general method for nonlinear equations gives the exact initial slope which gives the solution of the BVP at the first Newton iteration when $u'(0) = s = 0$ is chosen. Is this true for any other values of s?

Question 4. Use the shooting method to solve the BVP

$$u'' = 100u, \quad u(0) = 1, \quad u(3) = \epsilon + \cosh 30.$$

If the linearly independent solutions

$$u_0(x) = \cosh 10x, \quad u_1(x) = \frac{1}{10}\sinh 10x$$

are employed to obtain $u(x) = u_0(x) + \gamma u_1(x)$, show that

$$\gamma = 10\epsilon/\sinh 30.$$

Is there a numerical difficulty in carrying out the numerical solution for small $|\epsilon| \ll \cosh 30$?

Question 5. (Optional) Implement the nonlinear shooting method to approximately solve

$$y'' = y^3 - yy' \quad \text{with b. c. } y(1) = 1/2, \ y(3) = 1/4$$

for $x \in [1, 3]$. This can be implemented relatively simply in Matlab using a build in IVP solver such as ode45 and the Secant method for the root finding component. Note that the exact solution is $y(x) = 1/(1 + x)$.

Question 6. For the problem

$$(pu')' + qu = f, \quad u(a) = \alpha, \quad u(b) = \beta,$$

approximated by

$$\frac{p(x_{j+\frac{1}{2}})u_{j+1} - \left[p(x_{j+\frac{1}{2}}) + p(x_{j-\frac{1}{2}})\right] u_j + p(x_{j-\frac{1}{2}})u_{j-1}}{h^2} + q(x_j)u_j = f(x_j),$$

$$j = 1, \dots, n, \quad u_0 = \alpha, \quad u_{n+1} = \beta,$$

show that the local truncation error is $\mathcal{O}(h^2)$ provided $u(x)$ and $p(x)$ have four derivatives.

[Note: $p(x_{j\pm\frac{1}{2}}) = p(x_j) \pm \frac{h}{2}p'(x_j) + \frac{1}{8}h^2 p''(x_j) \pm \frac{h^3}{48}p'''(x_j) + \mathcal{O}(h^4)$.]