Numerical Solution of Differential Equations II. QS 3

Question 1. Show that for each $r, s = 1, \ldots, n$, the vector

$$v^{rs} = (v_{11}^{rs}, v_{12}^{rs}, \ldots, v_{1n}^{rs}; v_{21}^{rs}, v_{22}^{rs}, \ldots, v_{2n}^{rs}; \ldots; v_{n1}^{rs}, v_{n2}^{rs}, \ldots, v_{nn}^{rs})^T$$

with $v_{jk}^{rs} = \sin \frac{jr\pi}{n+1} \sin \frac{ks\pi}{n+1}$ is an eigenvector of the 5-point finite difference matrix associated with

$$\frac{u_{j-1,k} + u_{j+1,k} - 4u_{j,k} + u_{j,k-1} + u_{j,k+1}}{h^2} = f(x_j, y_k),$$

with corresponding eigenvalue $-\frac{1}{h^2}\left[4 - 2\cos \frac{r\pi}{n+1} - 2\cos \frac{s\pi}{n+1}\right]$.

Question 2. Apply the maximum principle using a comparison function to show that $|u_j - u(x_j)| = \mathcal{O}(h^2)$ where $u(x)$ is the solution to

$$L(u) = u'' = f(x) \quad \text{on} \quad [a, b], \quad u(a) = \alpha, \quad u(b) = \beta,$$

and u_j is the solution to

$$L_h u_j = \frac{u_{j-1} - 2u_j + u_{j+1}}{h^2} = f(x_j) \quad \text{for} \quad j = 1, 2, \ldots, n$$

and $u_0 = \alpha$, $u_{n+1} = \beta$ with $x_j = a + jh$ for $h = (b - a)/(n + 1)$.

Question 3. Suppose Ω is the right-angled isosceles triangle

$$\{(x, y) \; : \; 0 \le x \le 1, \;\; 0 \le y \le 1 - x\}$$

and $\delta\Omega$ its boundary. Use a regular grid on this domain with $x_0 = 0$, $x_j = jh$, $(n + 1)h = 1$, $y_0 = 0$, $y_k = kh$ and apply the 5-point finite difference formula to approximate the solution of $-\nabla^2 u = f$ on Ω with $u = 0$ on $\delta\Omega$. How many rows of the coefficient matrix have only 2 non-zero entries? How many have 3, 4 non-zero entries? What is the structure of the coefficient matrix with Lexicographic ordering? (This is not so easy as for a square! Do it for $n = 5$ if you prefer.) Is this matrix Irreducibly Diagonal Dominant?

Question 4. Verify that the matrix obtained for central difference approximation of

$$u'' + qu = f \quad \text{on} \quad [a, b], \quad u(a) = \alpha, \quad u'(b) = \beta,$$

where $q \in \mathbb{R}$ is a constant, is Irreducibly Diagonally Dominant for every (small) mesh size h if and only if $q \le 0$.