Numerical Solution of Differential Equations II. QS 6

Question 1. Prove statements 1,2, and 4 of the following result. Statement 3 (also known as the Crandall-Tartar lemma) was established during the lecture. For statement 4 you are allowed to use statement 3.

Theorem 1. *For a monotone scheme, it holds that*

(1) $U_j^n \le V_j^n \Rightarrow G(U^n)_j \le G(V^n)_j$, *for all* j.
(2) *The produced solution satisfies a maximum principle*

$$\min_{i \in S_j} U_i^n \le G(U^n)_j \le \max_{i \in S_j} U_i^n,$$

where S_j *stands for the stencil around* x_j.
(3) *The scheme is* L_1-*contractive.*
(4) *The scheme is Total Variation Diminishing (TVD).*

Question 2. Now the idea is to follow the steps of the sketch given during the lecture (using the Kruzhkov entropy pair), to complete the proof of the following result.

Theorem 2. *The solution produced by a monotone scheme satisfies all entropy conditions.*

Question 3. Consider the linear problem

$$u_t + u_x = 0.$$

For the Lax-Friedrichs scheme, derive a condition (on $\lambda = k/h$) for the method to be monotone. Can a similar result be established for the Lax-Wendroff scheme?