# B5.6: Nonlinear Systems-Sheet 1

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**Q1** Consider the system $\dot{x} = Ax$ where $x \in \mathrm{R}^3$ and

$$A = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 2 & 0 & 6 \end{bmatrix}. \tag{1}$$

Without solving the system, find the stable, unstable and center subspaces and sketch the phase portrait.

**Q2 (invariant set)** The system

$$\begin{aligned} \dot{x} &= -x, & (2) \\ \dot{y} &= -y + x^2, & (3) \\ \dot{z} &= z + x^2 & (4) \end{aligned}$$

defines a flow $\varphi_t : \mathbb{R}^3 \to \mathbb{R}^3$. Show that the set $S = \{(x, y, z) \in \mathbb{R}^3 | \, z = -x^2/3\}$ is an invariant set of this flow. Sketch this set in the phase space and identify other interesting orbits (such as fixed points).

**Q3 (attracting set)** The system

$$\begin{aligned} \dot{x} &= -y + x(1 - z^2 - x^2 - y^2), & (5) \\ \dot{y} &= x + y(1 - z^2 - x^2 - y^2), & (6) \\ \dot{z} &= 0 & (7) \end{aligned}$$

defines a flow $\varphi_t : \mathbb{R}^3 \to \mathbb{R}^3$. Show that the union of the unit sphere with the portions of the z-axis outside the sphere is an attracting set for this flow. Find its domain of attraction. (Hint: rewrite the system in cylindrical coordinates).

**Q4 (attracting set)** Consider the system

$$\begin{aligned} \dot{r} &= r(1 - r), & (8) \\ \dot{\theta} &= \sin^2 \frac{\theta}{2}, & (9) \end{aligned}$$

where $(r, \theta)$ are the usual polar coordinates of a point in the plane. Show that this system has two fixed points. Show that the fixed point $(x = 1, y = 0)$ is the $\omega$-limit set of almost all initial conditions. That is $\varphi_t(x_0) \to (1, 0)$ for all initial conditions $x_0 \neq (0, 0)$. Despite that, show that $(1, 0)$ is not stable. Is it an attracting set? Is the unit circle an attracting set? Find the domain of attraction (if any).

**Q5* (the variational equation and its adjoint):** Consider the system $\dot{x} = f(x)$, where $x \in \mathbb{R}^n$ and $f(x)$ is a $C^1$-vector field. Let $\bar{x} = \bar{x}(t)$ be a particular solution and consider the linear system obtained by looking at a nearby solution $x(t) = \bar{x}(t) + \varepsilon u$:

$$\dot{u} = Df(\bar{x})u$$

$(i)$ Show that $\dot{\bar{x}}(t)$ is a solution of the linear system, and it represents the tangent vector along the orbit.
$(ii)$ For planar flows ($n = 2$), use this property to give a complete solution of this system of linear equations.