1. Linear Systems

1.1 Fundamental theorems.

General problem:

$$(1)\quad \begin{cases} \dot{x} = Ax & x \in \mathbb{R}^n, \; A \in \mathcal{M}_n(\mathbb{R}) \\ x_0 = x(t_0) \end{cases}$$

$\mathcal{M}_n(\mathbb{R})$: $n \times n$ matrices w/ coeff. in $\mathbb{R}$.

If $n=1$ $\quad \dot{x} = ax \Rightarrow x(t) = e^{ta} x_0$

In general "$x = e^{tA} x_0$" but what is e^{tA} ?

<u>Definition</u>: $A \in \mathcal{M}_n(\mathbb{R})$, $t \in \mathbb{R}$

$$(2) \Rightarrow e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$$

This series is absolutely, uniformly conv $\forall \, |t| < T$

<u>Thm</u> $\exists\, !$ solution of system (1):

$$x = e^{tA} x_0$$

<u>Property</u> If $C = BAB^{-1} \Rightarrow e^C = B e^A B^{-1}$

$\Rightarrow$ If $C = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$, $A = BCB^{-1}$

$$\Rightarrow e^{tA} = B \,\mathrm{diag}(e^{\lambda_i t})\, B^{-1}$$

Examples in $\mathbb{R}^2$

$$A \in \mathcal{M}_2(\mathbb{R}) \Rightarrow \begin{cases} \dot{x}_1 = a_{11}x_1 + a_{12}x_2 \\ \dot{x}_2 = a_{21}x_1 + a_{22}x_2 \end{cases} \quad a_{ij} \in \mathbb{R}$$

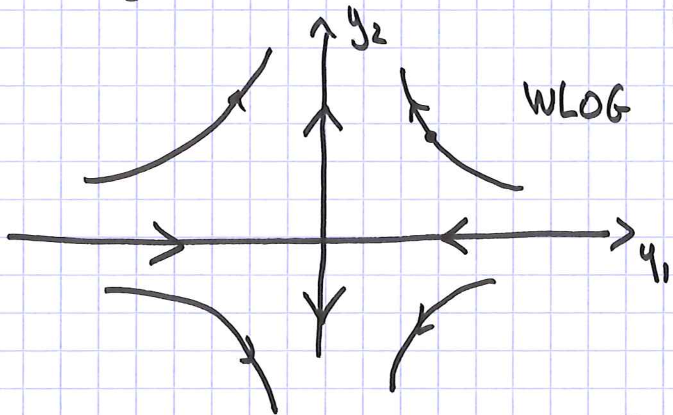
$$\Leftrightarrow \dot{x} = Ax$$

$\Rightarrow \exists B \in GL(\mathbb{R})$ s.t. $C = BAB^{-1}$ where C has one of the following form based on $\lambda_1, \lambda_2 \in \mathrm{Spec}(A)$ (The eigenvalues of A)

1. $\lambda_1, \lambda_2 \in \mathbb{R}$

1.a $\lambda_1 \lambda_2 < 0 \Rightarrow C = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

Define $y = Bx$ then $\begin{cases} \dot{y}_1 = \lambda_1 y_1 \\ \dot{y}_2 = \lambda_2 y_2 \end{cases}$



WLOG $\lambda_1 < 0 < \lambda_2$

SADDLE

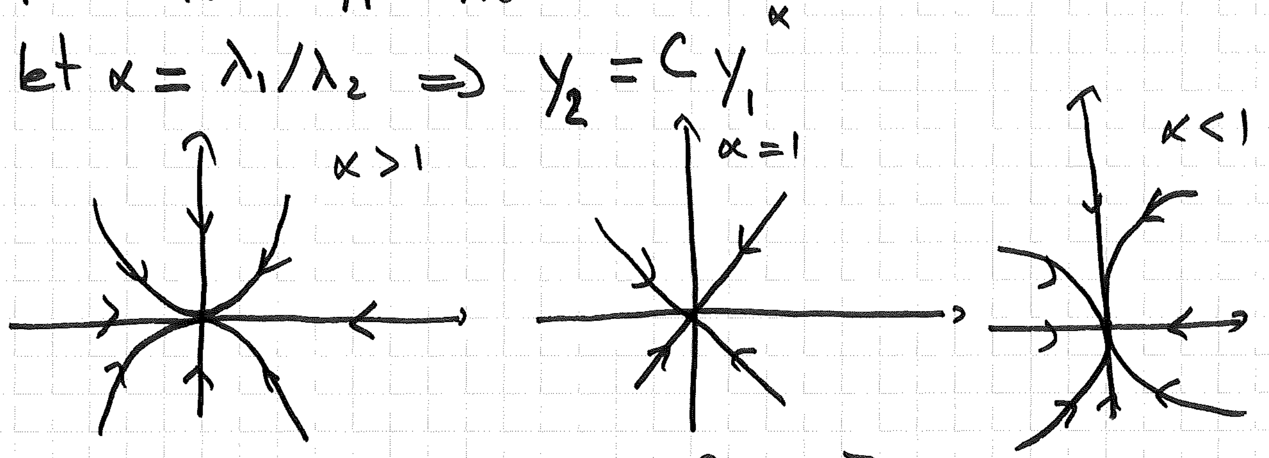
What does this picture mean?
Phase portrait. $e^{tC} \Leftrightarrow$ curves in $\mathbb{R}^2$

1.b $\lambda_1, \lambda_2 > 0 \quad \lambda_1 \neq \lambda_2 \Rightarrow C = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ WLOG $\lambda_1 < 0$, $\lambda_2 < 0$

$$\dot{y}_i = \lambda_i y_i \Rightarrow y_i = y_{i0}\, e^{\lambda_i t}$$

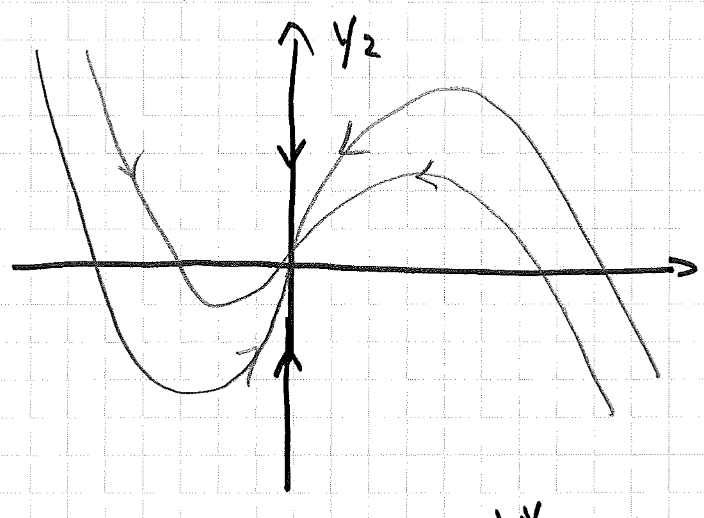
let $\alpha = \lambda_1 / \lambda_2 \Rightarrow y_2 = C\, y_1^{\alpha}$

NODE



1.c $\lambda_1 = \lambda_2 = \lambda$ but $C = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

$$\Rightarrow \begin{cases} y_2 = y_{20}\, e^{\lambda t} \\ y_1 = y_{10}\, e^{\lambda t} + y_{20}\, t e^{\lambda t} \end{cases}$$

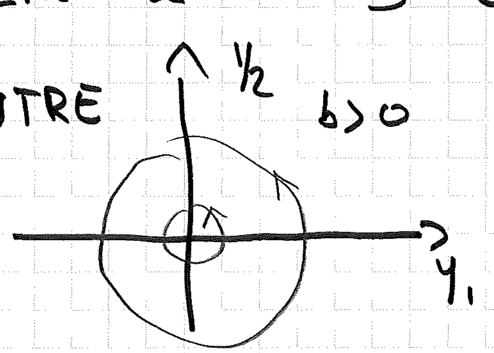


NODE

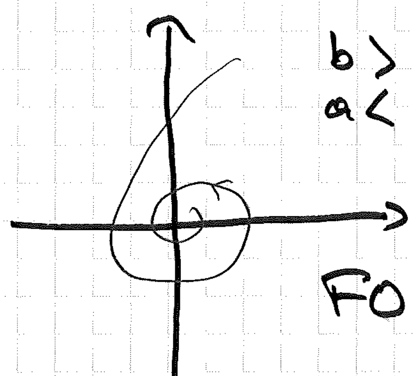
2. $\lambda_1, \lambda_2 \in \mathbb{C} \Rightarrow \lambda_1 = \lambda_2^* \quad \lambda_1 = a + ib$

2.a $a = 0 \Rightarrow C = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}$

CENTRE



2.b $a \neq 0 \Rightarrow C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$



$b > 0$
$a < 0$

FOCUS

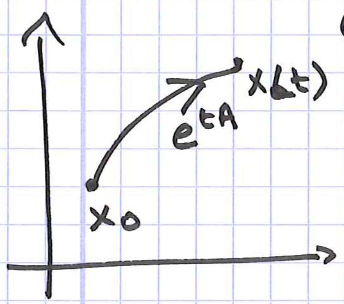
1.2 Linear flows

$\dot{x} = Ax \qquad x \in \mathbb{R}^n, \ A \in \mathcal{M}_n(\mathbb{R})$

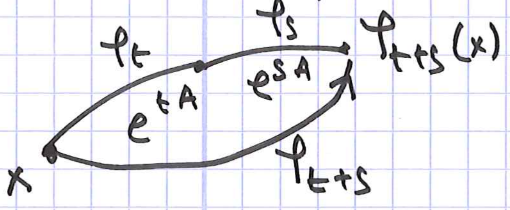
Solution $\quad x(t) = e^{tA} x_0$

Geometrically e^{tA} is a map

$$e^{tA} : \mathbb{R}^n \to \mathbb{R}^n \qquad \underline{\text{linear flow}}$$



Let $\varphi_t = e^{tA} \Rightarrow \varphi_t(x_0) = e^{tA} x_0 = x(t)$



Properties: $\begin{cases} \varphi_0 = \mathbb{1} \\ \varphi_{t+s}(\underline{x}) = \varphi_t(\varphi_s(x)) \quad \forall x \in \mathbb{R}^n \end{cases}$

Important class of flows:

Set of eigenvalues of A: $\{\lambda_1, \dots, \lambda_n\} = \mathrm{Spec}(A)$

__Def.__: If A is s.t. $\mathrm{Re}(\lambda) \neq 0 \ \forall \lambda \in \mathrm{Spec}(A)$
then the flow e^{tA} is __hyperbolic__
the system $\dot{x} = Ax$ is __hyperbolic__

__Def.__ Let $E \subset \mathbb{R}^n$, E is an invariant set of φ if $\varphi_t(E) \subset E \ \forall t$

__Example__ $\quad E = \mathrm{span}(v)$ where v is the eigenvec of eigenvalue λ is an invariant set

· <u>Indeed</u>

$$E = \text{span}(v) = \{cv,\ c \in \mathbb{R}\}$$

$$\varphi_t(cv) = e^{tA} cv = c e^{At} v$$

$$= c v e^{\lambda t} = \tilde{c} v \in E \quad \forall c, v$$

. We can now construct 3 subspaces depending on the real part of the eigenvalues.

. Here we assume that A is semi-simple in $\mathbb{C}^n$ (it can be diagonalized over $\mathbb{C}$).

$$\{\lambda_1, \dots, \lambda_n\} = \text{Spec}(A)$$

$$\begin{cases} \lambda_j = a_j + i b_j & j = 1, \dots, n \\ w_j = u_j + i v_j & \text{eigenvectors}. \end{cases}$$

$$\Leftrightarrow \quad A w_j = \lambda_j w_j \quad j = 1, \dots, n.$$

Then, we <u>define</u>:

$$E^s = \text{Span}\{u_j, v_j \mid a_j < 0\}$$

$$E^c = \text{Span}\{u_j, v_j \mid a_j = 0\}$$

$$E^u = \text{Span}\{u_j, v_j \mid a_j > 0\}$$

The stable (s), center (c), unstable (u) linear subspaces.

$\text{Dim.}(E^s) = n_s$

$\text{Dim}(E^c) = n_c$ $\qquad n_s + n_c + n_u = n.$

$\text{Dim}(E^u) = n_u$

NB: If A is not semisimple then w_j are taken as the generalized eigenvectors

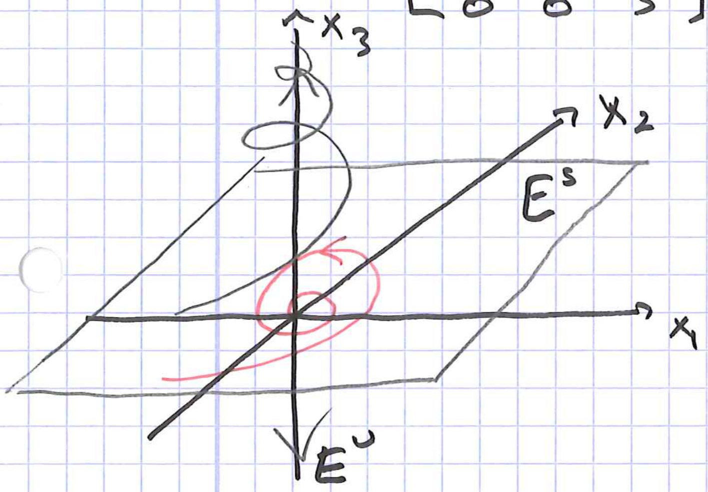
Examples

1) $A = \begin{bmatrix} -2 & -1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$$\begin{cases} w_1 = u_1 + i v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ \lambda_1 = -2 + i \end{cases}$$

$$\begin{cases} w_2 = u_1 - i v_1 \\ \lambda_2 = -2 - i \end{cases}$$

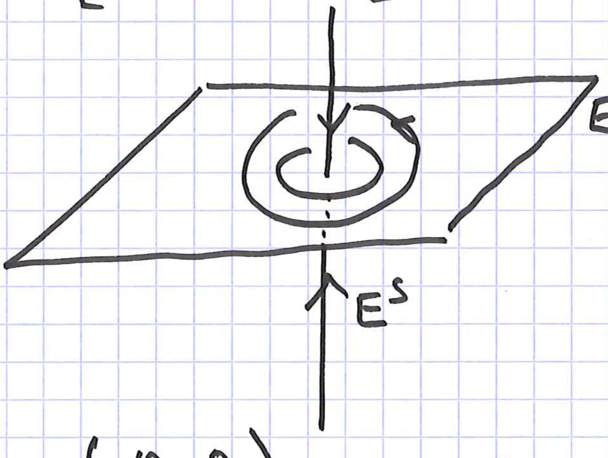
$$\left\{ w_3 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \ \lambda_3 = 3 \right.$$



2) $A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

$\lambda_1 = i \quad \lambda_2 = -i \quad \lambda_3 = -2$

$u_1 = e_1 \quad v_1 = e_2 \quad v_3 = e_3$



$E^c = \mathrm{Span}\{e_1, e_2\}$

3. $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \lambda_1 = \lambda_2 = 0 \quad E^c = \mathbb{R}^2$