B5.6: Nonlinear Systems-Sheet 1 (solutions)

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January 27, 2019

Q1 To determine the subspaces of the matrix

$$A = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 2 & 0 & 6 \end{bmatrix}, \tag{1}$$

we find its eigenvalues and eigenvectors, which are:

$$\begin{aligned} \lambda_1 &= 6 \quad \mathbf{v}^{(1)} = (0,0,1)^T, \\ \lambda_2 &= 2i \quad \mathbf{v}^{(2)} = i(0,10,-1)^T + (10,0,-3)^T, \\ \lambda_3 &= -2i \quad \mathbf{v}^{(3)} = -i(0,10,-1)^T + (10,0,-3)^T. \end{aligned}$$

Therefore, the stable E^s, unstable E^u, and center E^c subspaces of (1) are:

$$\begin{aligned} E^s &= \emptyset, \\ E^u &= \mathrm{span}\{(0,0,1)^T\}, \\ E^c &= \mathrm{span}\{(0,10,-1)^T,\ (10,0,-3)^T\}; \end{aligned}$$

that is, the empty set, the z-axis, and the plane specified by $z = -3x/10 - y/10$.

Q2 (invariant set) For the system:

$$\dot{x} = -x, \tag{2}$$
$$\dot{y} = -y + x^2, \tag{3}$$
$$\dot{z} = z + x^2 \tag{4}$$

we define a quantity $Q(t) = z + x^2/3$ and find, upon differentiating:

$$\dot{Q} = z + x^2/3,$$

which vanishes only if $z = -x^2/3$. Therefore, $z = -x^2/3$ is an invariant set of the system. The other invariant set is the fixed point at (0, 0, 0), which, by linearization, is a saddle node because it has eigenvalues and eigenvectors given by:

$$\begin{aligned}
\lambda_1 &= 1 \quad \mathbf{v}^{(1)} = (0,0,1)^T, \\
\lambda_2 &= -1 \quad \mathbf{v}^{(2)} = (1,0,0)^T, \\
\lambda_3 &= -1 \quad \mathbf{v}^{(3)} = (0,1,0)^T.
\end{aligned}$$

The phase portrait of the system is shown in Fig. 2.

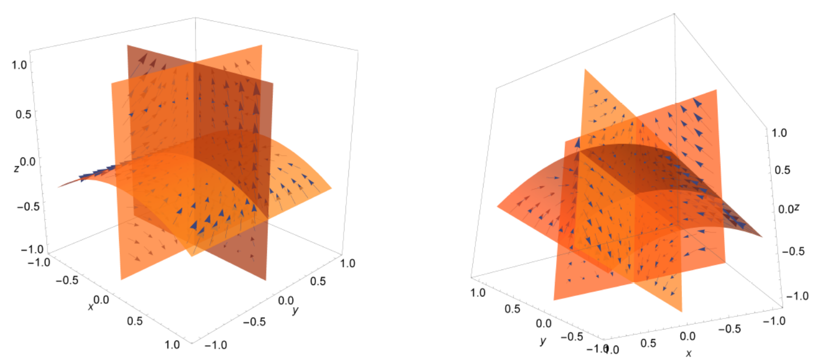

Figure 1: Phase portrait of (2)-(4) on the planes $x = y = 0$ and $z = -x^2/3$.

Q3 (attracting set) We consider the system:

$$\begin{aligned}
\dot{x} &= -y + x(1 - z^2 - x^2 - y^2), && (5)\\
\dot{y} &= x + y(1 - z^2 - x^2 - y^2), && (6)\\
\dot{z} &= 0 && (7)
\end{aligned}$$

We introduce the radius in cylindrical coordinates $r^2 = x^2 + y^2$ and differentiate with respect to time:

$$r\dot{r} = x\dot{x} + y\dot{y}, \quad \Longrightarrow \quad \dot{r} = r(1 - r^2 - z^2),$$

which suggests $r = 0$ (i.e z-axis) is the α-limit set (asymptotically unstable in this case) whilst $r = \sqrt{1 - z^2}$ is the ω-limit set (asymptotically stable in this case). Each circle is a level set in $z \in [-1, 1]$, which together form a unit sphere that is an attracting set (i.e. the orbits within a shell of radius $\rho = \sqrt{1-r^2-z^2} \in [1-\varepsilon,\, 1+\varepsilon] \quad \forall \varepsilon > 0$ will tend to the unit sphere as $t \to \infty$). Defining the unit sphere as S^1, the domain of attraction would be $D(S^1) = \mathbb{R}^2 \times [-1,1] \setminus \{0\}$, given that $\dot{z} = 0$. In turn, the domain of attraction of the part of z-axis with $|z| > 1$ is given by $\mathbb{R}^2 \setminus D(S^1)$.

We note that there does not exist a dense orbit Γ on the unit sphere such that, $\forall\, \Gamma$ also on the unit sphere, there are points $x \in \Gamma$ and $\tilde{x} \in \Gamma$ satisfying $|x - \tilde{x}| < \varepsilon \quad \forall \varepsilon > 0$. Orbits on the unit sphere are confined to circles at a particular z and will, therefore, never come arbitrarily close together.

Q4 (attracting set) We study the system:

$$\dot{r} = r(1-r), \quad \dot{\theta} = \sin^2\frac{\theta}{2}, \tag{8}$$

We note that fixed points occur at $r = \{0, 1\}$ and $\theta = 2\pi n$ for $n \in \mathrm{Z} \cup \{0\}$, which correspond to two fixed points in Cartesian coordinates, $(x,\, y) = (0,\, 0)$ and $(1,\, 0)$. The radius $r = 0$, or the fixed point $(0, 0)$, is the α-limit set whilst $r = 1$ is asymptotically stable. Due to (8), the trajectories proceed around the unit circle in an anti-clockwise direction and asymptotically approach $(1,\, 0)$. Therefore, $(1, 0)$ is the ω-limit point in the system for all initial conditions except the fixed point at the origin (i.e. $x_0 \neq 0$).

Despite this, however, $\theta = 0$ is unstable according to (8). Perturbing the state by $\theta = \varepsilon > 0$ for $\varepsilon \ll 1$ will result in the trajectory proceeding anti-clockwise around the unit circle until it reaches $(1, 0)$ again.

The unit circle is an attracting set because it is an invariant set and all trajectories that are within the radius $r \in [1-\varepsilon,\, 1+\varepsilon]$, for $\varepsilon > 0$, will end up within the set. The fixed point $(1, 0)$ is also an attracting set since the same neighborhood as above satisfies the definition. The domain of attraction for both the unit circle and $(1,\, 0)$ is $D(r = 1) = D((1, 0)) = \mathbb{R}^2 \setminus \{0\}$.

Q5* (the variational equation and its adjoint): (i) We are asked to show that $u(t) = \dot{\bar{x}}(t)$ is a solution of the linear equation

$$\dot{y} = Df(\bar{x})y, \tag{9}$$

where $\bar{x}(t)$ is a particular solution of $\dot{x} = f(x)$. We note that:

$$\begin{aligned} \dot{\bar{x}} &= f(\bar{x}), \\ \implies \ddot{\bar{x}} &= Df(\bar{x})\dot{\bar{x}}, \end{aligned}$$

which is (9) for $y = u = \dot{\bar{x}}$, as required. The vector $\dot{\bar{x}}(t)$ is tangential to the flow by construction and is analogous to the velocity of a particle traveling on a curve whose position is given by $\bar{x}(t)$.

(ii) For planar flows, we require two linearly independent solutions to (9). From (i) we know that one of this solutions is $u(t) = \dot{\bar{x}}(t)$. To determine the other solution, which we will call $v(t)$, we suppose it has the general form:

$$v(t) = g(t)\tau(t) + h(t)n(t), \tag{10}$$

where $\tau(t) = \dot{\bar{x}}(t)$ is a vector tangential to the flow traced out by $\bar{x}(t)$ and $n(t) = \tau(t)^{\perp}$ is the corresponding normal. Note that, for a prescribed $\bar{x}(t)$, both $\tau(t)$ and $n(t)$ are known and they are orthogonal to each other. Given that $v(t)$ is a solution to linear system (9), we differentiate (10) with respect to t to obtain:

$$\dot{g}\tau + g\dot{\tau} + \dot{h}n + h\dot{n} = Df(\bar{x})g\tau + Df(\bar{x})hn, \implies \dot{g}\tau + \dot{h}n + h\dot{n} = Df(\bar{x})hn, \tag{11}$$

where the latter step results from the definition of $\tau(t) = \dot{\bar{x}}(t)$ and (9).

Multiplying (11) by n, we use orthogonality to obtain:

$$\dot{h} = \frac{h}{|n|^2}(nDf(\bar{x})n - \dot{n}n) = K_1(t)h, \implies h(t) = c_1 \exp\left(\int K_1(t)\,dt\right), \tag{12}$$

for $c_1 \in \mathbb{R}$. Note that $K_1(t)$ is completely known for a given $\bar{x}(t)$. Taking (11) and multiplying through by τ, we now obtain:

$$\dot{h} = \frac{h}{|\tau|^2}(\tau Df(\bar{x})n - \dot{n}\cdot\tau) = K_2(t)h, \implies h(t) = \int K_2 h\,dt + c_2, \tag{13}$$

for $c_2 \in \mathbb{R}$. Similarly, $K_2(t)$ is completely known. We obtain the explicit form of $v(t)$ by combining (10), (12) and (13) above, so that the complete solution of (9) is

$$y(t) = a_1 u(t) + a_2 v(t)$$

for $a_1,\, a_2 \in \mathbb{R}$.