B5.6: Nonlinear Systems-Sheet 3

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Q1 (The complex Landau equation)

$$\dot{z} = az - b|z|^2 z,$$

arises in nonlinear stability theory. Here $z(t)$ is complex-valued and a, b are complex numbers (assume that $\mathcal{R}e\, a > 0$). Write the equation as a system of two real equations for $r(t)$ and $\theta(t)$, where $z = re^{i\theta}$. Discuss the existence of periodic solutions in terms of the constants a, b.

Q2 (The glider) A simple model for the motion of a glider is given by the equations

$$\begin{aligned}\dot{y} &= -\sin\theta - ay^2,\\ \dot{\theta} &= y - \frac{\cos\theta}{y},\end{aligned}$$

where y is the velocity, θ is the angle between the glider and the horizontal, and a is the ratio of the drag coefficient to the lift coefficient. For $a = 0$ show that $y^3 - 3y\cos\theta$ is a conserved quantity and sketch the phase portrait. Interpret your result (What does the glider do? What is its path?).
*For $a > 0$ (positive drag), linearise the system around its fixed points and discuss the stability. Again, interpret this result in terms of its motion.

Q3 (Non-wandering sets) A point p is *non-wandering* for a flow φ if, for any neighbourhood U of p, there exist arbitrarily large times t, such that $\varphi_t(U) \cap U \neq \emptyset$. A set Ω is *non-wandering* if all points $p \in \Omega$ are non-wandering.

Find the non-wandering sets for the following flows:

(i) $\dot{\theta} = \mu - \sin\theta$ where $\theta \in S^1$ (the unit circle). Hint: consider cases $\mu < 1$, $\mu = 1$ and $\mu > 1$.

(ii) $\ddot{\theta} + \sin\theta = 1/2,\ \theta \in S^1$.

Q4 (Gradient vector fields) Let $V(x)$ be a $C^r (r \geq 1)$ function of $x \in \mathbb{R}^n$. A *gradient vector field* is defined as:

$$\dot{x} = -\nabla V(x).$$

Show that the non-wandering set of a gradient vector field on $\mathbb{R}^2$ contains only fixed points and that no periodic or homoclinic orbits are possible. Hint: Use $V(x)$ as a Lyapunov function.

Q5 (The Lorenz system) By using ideas similar to Lyapunov's function method, show that all trajectories of the Lorenz system

$$\begin{aligned}\dot{x} &= \sigma(y - x),\\ \dot{y} &= \rho x - xz - y,\\ \dot{z} &= xy - \beta z.\end{aligned}$$

eventually enter and remain inside a large sphere S of the form $x^2 + y^2 + (z - \rho - \sigma)^2 = C$, for C sufficiently large.