B5.6: Nonlinear Systems-Sheet 4

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February 9, 2019

Q1 (limit cycles) Discuss stability of the equilibria and limit cycles of

$$\dot{x} = -y + xf(\sqrt{x^2 + y^2}), \quad (1)$$

$$\dot{y} = x + yf(\sqrt{x^2 + y^2}), \quad (2)$$

where $f(r) = \sin r$.

Q2 (a bead on a wire) A bead is free to slide without friction on a circular wire hoop. The hoop spins about its vertical axis with angular velocity ω. The equation governing the position $\theta(t)$ (measured from the bottom of the hoop) is

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta - w^2 \sin\theta\cos\theta = 0.$$

(i) Discuss the behaviour of this system, as ω increases from zero, from the point of view of bifurcation theory.

(ii) Write down an energy integral for the system. Find the smallest constant $V > 0$ (in terms of the parameters) such that, if initially $\theta = \pi/2$, $|\dot{\theta}| > V$, then the bead will continually encircle the hoop in one direction.

(iii)$*$ What happens if a linear damping is added to the system (that is adding $-\mu\dot{\theta}$ to the equation's LHS with $\mu > 0$)? (NB: There is a real zoo of possible bifurcations in this system. A simple and good starting point is to find the critical rotation speed at which $\theta = 0$ becomes unstable. Describe this bifurcation).

Q3 (centre manifold to fourth order) Consider the system

$$\dot{x} = y - x - x^2, \quad (3)$$

$$\dot{y} = \mu x - y - y^2. \quad (4)$$

Find the value of μ for which there is a bifurcation at the origin. Find the evolution equation on the extended centre manifold correct to third order and determine the type of bifurcation.

Q4 (a 1D map) Consider the map

$$x_{n+1} = (1 + \mu)x_n - \mu x_n^2 = f(x_n, \mu), \quad (5)$$

for $\mu \geq 0$.

(i) Find the fixed points and analyse their stability.

(ii) Find the period-2 cycles and analyse their stability.

(iii) Draw the bifurcation diagram in the (μ, x)-plane.

(iv)$*$ Verify your results by computing (numerically) the full bifurcation diagram of x_n for $0 < \mu \leq 3$.

Q5 (stability of periodic orbits) Consider a 1D map

$$x_{n+1} = f(x_n),$$

and assume that it supports a p-periodic orbit $\{x_1, x_2, ..., x_p\}$ such that $x_i \neq x_j, \quad \forall i, j \in \{1, ..., p\}$ with $i \neq j$, and $x_{p+1} = x_1$. Show that the stability of this orbit is determined by the multiplier

$$\lambda = \prod_{i=1}^{p} f'(x_i),$$

whenever $|\lambda| \neq 1$.