Nonlinear Systems-lecture notes 4

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4 Local bifurcations of continuous and discrete dynamical systems

The material of this chapter is covered in the following books:

- L. Perko, *Differential Equations and Dynamical Systems* (Second edition, Springer, 1996). Paragraphs **4.1-4.2, 4.4-4.5**.
- Guckenheimer and Holmes, *Nonlinear Oscillations, Dynamical Systems* (Springer, 1983). Paragraphs **1.7-1.9, 3.1, 3.4-3.5**.
- Y. A. Kuznetsov, *Elements of Applied Bifurcation Theory* (Second edition, Springer, 1998). Paragraphs **2.1, 2.3, 3.1-3.5, 4.1-4.5, 4.9**.
- S. H. Strogatz, *Nonlinear Dynamics and Chaos with Applications to Physics, Biology, Chemistry and Engineering* (Westview Press, 2000). Paragraphs **8.0-8.2, 10.1-10.6**.

4.1 Equivalence and structural stability of dynamical systems

Definition 4.1. Two dynamical systems

$$\dot{x} = f(x),\ \ x \in \mathbb{R}^n \tag{1}$$

and

$$\dot{y} = g(y), y \in \mathbb{R}^n \tag{2}$$

are called *topologically equivalent* if there exists a homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ mapping orbits of the first system onto orbits of the second, while preserving the direction of time. Namely, for any $x \in \mathbb{R}^n$ there exist two times t_1, $t_2 \in \mathbb{R}$ such that

$$h(\varphi^f_{t_1}(x)) = \varphi^g_{t_2}(h(x)) \tag{3}$$

holds.

Example 4.1 (smoothly equivalent systems)**.** Suppose $h : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism (differentiable and having differentiable inverse) such that

$$f(x) = [\nabla h(x)]^{-1} g(h(x)) \tag{4}$$

holds between the right-hand sides of (1) and (2). Then systems (1) and (2) are topologically equivalent.

Proof: By differentiating $y = h(x)$ one obtains

$$\dot{y} = \nabla h(x)\dot{x} = \nabla h(x) f(x) = g(h(x)), \tag{5}$$

where in the last equality we used (4). Hence (3) holds. □

Let us consider a fixed point $x = x_0$ of system (1), which is mapped into the fixed point $y_0 = h(x_0)$ of system (2). Assume additionally that systems (1) and (2) are smoothly equivalent. By taking differential of (4) at $x = x_0$ and using the chain rule one obtains

$$Df(x_0) = [\nabla h(x_0)]^{-1} Dg(h(x_0))\nabla h(x_0)$$

From this we note that Jacobians of (1) and (2) are related through a linear transformation:

$$Df(x_0) = B^{-1} Dg(h(x_0))B \quad \text{with matrix} \quad B = Dh(x_0).$$

Therefore, we conclude that

$$\operatorname{spec}\{Df(x_0)\} = \operatorname{spec}\{Dg(y_0)\},$$

which implies that stability properties of (1) at $x = x_0$ and (2) at $y_0 = h(x_0)$ are the same. In particular, one has

$$\begin{aligned} \dim W^s_{loc}(x_0) &= \dim W^s_{loc}(h(x_0)), \\ \dim W^u_{loc}(x_0) &= \dim W^u_{loc}(h(x_0)). \end{aligned}$$

Example 4.2 (orbitally equivalent systems)**.** Let $f(x) = \mu(x)g(x)$, $x \in \mathbb{R}^n$ with function $\mu > 0 : \mathbb{R}^n \to \mathbb{R}$. Then systems (1) and (2) are called orbitally equivalent. In particular, such systems are topologically equivalent with

$$y = h(x) = x, \quad \varphi^f_{t_1}(x) = \varphi^g_{t_2}(x).$$

One can see that the orbits of systems (1) and (2) in this case are the same, but the velocities $\dot{x}$ and $\dot{y}$ along them differ by a factor $\mu(x)$.

The notion of topological equivalence can be considered also just locally in the vicinity of a fixed point.

Definition 4.2. (1) is called locally topologically equivalent in a neighborhood $U(x_0)$ of a fixed point $x = x_0$ to (2) in a neighborhood $V(y_0)$ of a fixed point $y = y_0$, if there exists a homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$:

(i) $h : U(x_0) \to \mathbb{R}^n$,

(ii) $h(x_0) = y_0$

(iii) h maps orbits in the neighbourhood $U(x_0)$ into the corresponding orbits in the neighbourhood $V(y_0) = h(U(x_0)) \in \mathbb{R}^n$, while preserving the direction of time.

Example 4.3 (Node-focus equivalence)**.** Consider two linear systems

$$\begin{aligned} \dot{x_1} &= -x_1, \\ \dot{x_2} &= -x_2 \end{aligned} \tag{6}$$

and

$$\begin{aligned} \dot{x_1} &= -x_1 - x_2, \\ \dot{x_2} &= x_1 - x_2 \end{aligned} \tag{7}$$

The solutions to these systems can be found explicitly in the polar coordinates as

$$\rho_1(t) = \rho_0 e^{-t}, \quad \theta_1(t) = \theta_0,$$

and

$$\rho_2(t) = \rho_0 e^{-t}, \quad \theta_2(t) = \theta_0 + t,$$

respectively, for given initial data (ρ_0, θ_0). Origin is a stable node for system (6) and a stable spiral for (7) (see Fig. 1).

Let us consider special initial point $(1, 0)$ and calculate the time needed for both systems (6) and (7) to reach circle $\rho = \rho_0$ with $\rho_0 < 1$ starting from it. In both cases this time is given by $\tau(\rho_0) = -\ln(\rho_0)$. During $\tau(\rho_0)$ the polar angle for system (6) does not change, while for system (7) increases by amount $-\ln \rho_0$ (see Fig. 2). Therefore, there exists a map

$$h : \; \rho_1 = \rho_0, \quad \theta_1 = \theta_0 - \log \rho_1,$$

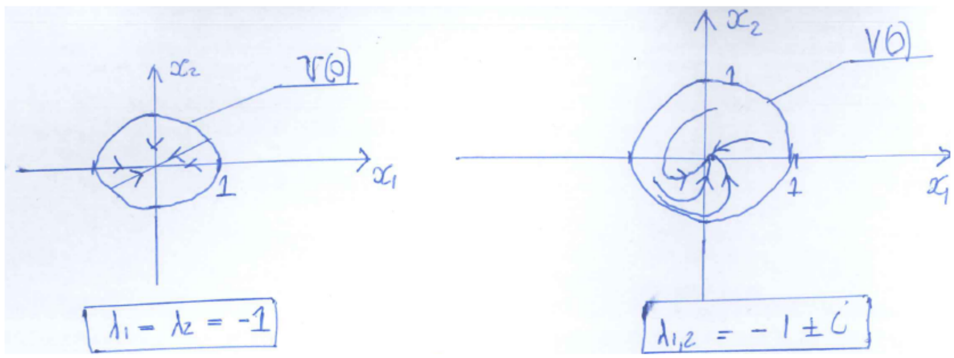

Figure 1: Phase plots of (6) (left) and (7) (right) in the unit ball neighborhood.

which maps orbits $(\rho_1(t), \theta_1(t))$ of (6) onto the corresponding ones $(\rho_2(t), \theta_2(t))$ of (7). By imposing an extra condition at the origin $h(0) = 0$ this map becomes a homeomorphism, i.e. we conclude that systems (6) and (7) are locally topologically equivalent in the respective neighbourhoods:

$$U(0) = V(0) = \{(x_1, x_2) : x_1^2 + x_2^2 \leq 1\} = \{(\rho, \theta) : \rho \leq 1\}$$

Nevertheless, (6) and (7) are neither *smoothly nor orbitally equivalent*. This is easy to see from Fig. 1, in particular eigenvalues of system (7) at the origin are different from those of (6).

The following theorem generalises the previous example.

Theorem 1. *The phase portraits of (6) and (7) near two hyperbolic fixed points $x = x_0$ and $y = y_0$ are locally topologically equivalent iff the numbers k_- and k_+ of eigenvalues of their Jacobians with $\Re e\lambda < 0$ and $\Im m\lambda > 0$ are the same.*

Sketch of the proof: *Step 1:* First, one needs to show that (1) is locally topologically equivalent to its linearisation at $x = x_0$:

$$\dot{\xi} = Df(x_0)\xi, \quad \xi = x - x_0.$$

This result for *hyperbolic fixed points* is given by Hartman-Grobman theorem (not examinable).

Step 2: Next, one needs to show topological equivalence of any two linear systems having the same numbers k_- and k_+.□

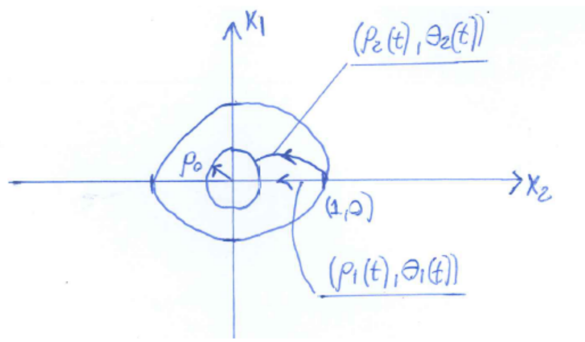

Figure 2: Two orbits to (6) and (7) starting at point $(0, 1)$ and reaching circle $\rho = \rho_0$ at $\tau(\rho_0) = -\ln(\rho_0)$.

4.2 Structural stability of dynamical systems

For any open bounded subset $E \subset \mathbb{R}^n$ let us define C^1 norm of $f : E \to \mathbb{R}^n$ as

$$||f||_1 = \max_{x \in E} |f(x)| + \max_{x \in E} ||Df(x)|| < \infty.$$

Definition 4.3. System

$$\dot{x} = f(x), \quad f \in C^1(E) \tag{8}$$

is called structurally stable in an open subset $E \subset \mathbb{R}^n$, if there exists $\varepsilon > 0$ such that $\forall g \in C^1(E)$ with

$$||f - g||_1 < \varepsilon$$

system (8) is topologically equivalent to

$$\dot{x} = g(x).$$

Example 4.4. System

$$\ddot{x} + \sin x = 0$$

is structurally unstable in the vicinity of the non-hyperbolic fixed point $x = 0$. Indeed, by adding a friction term $-A\dot{x}$ with $0 < A \ll 1$ and considering the damped oscillator

$$\ddot{x} + \sin x + A\dot{x} = 0, \tag{9}$$

the origin becomes a stable node. In contrast, the damped system is topologically equivalent for any pair of values $A = A_0$ and $A = A_1$ and, therefore, is structurally stable.

Example of a damped system (9) can be generalised to the following result about structural stability of *hyperbolic* fixed points.

Theorem 2. *Let $x = 0$ be a hyperbolic fixed point of system (8). Then for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $g \in C^1(E)$ with $||f - g||_1 < \delta$ there exists $y_0 \in B_\varepsilon(0)$ with $g(y_0) = 0$– the hyperbolic critical point of system*

$$\dot{y} = g(y).$$

Moreover, $Df(0)$ and $Dg(0)$ have the same number of eigenvalues with positive and negative real parts.

Example 4.5. Consider a dynamical system on a torus $S^1 \in \mathbb{R}^3$ (see Fig. 3):

$$\dot{x} = \omega_1, \quad \dot{y} = \omega_2 \tag{10}$$

with $(x, y) \in [0, 1)^2$. If $\omega_1/\omega_2 \in \mathbb{R} \setminus Q$ is an irrational number then any

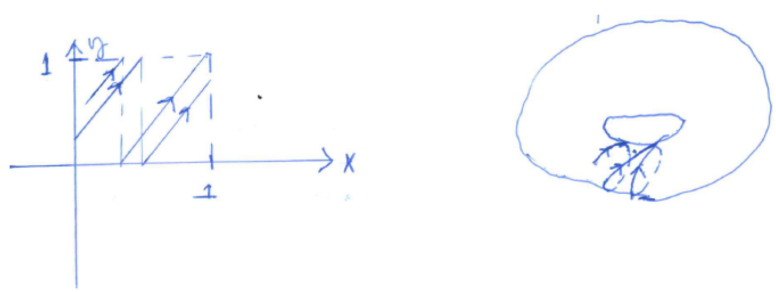

Figure 3: Left: trajectories of system (10) in $[0, 1)^2$ with horizontal and vertical sides being identified to each other, respectively. Right: The corresponding trajectories on the torus S^1.

initial data (x_0, y_0) generates an orbit dense everywhere in S^1. If in turn $\omega_1/\omega_2 \in Q$ is a rational number it is easy to check that any initial data (x_0, y_0) generates a periodic orbit. Therefore, there are infinitely many periodic orbits in this case. We can conclude that the system is structurally unstable for any $\omega_1/\omega_2 \in \mathbb{R}$, because there are small ε-order perturbations of ω_1/ω_2 which changes it from being rational to irrational and backwards.

The last example is connected with the notion of a non-wandering point.

Definition 4.4. A point $x \in \mathbb{R}^n$ in the phase space of system $\dot{x} = f(x)$ is called non-wandering if there exists a neighborhood $U(x) \in \mathbb{R}^n$ such that for arbitrary large t one has $\varphi_t(U) \cap U \neq 0$.

It easy to see that if $\omega_1/\omega_2 \in \mathbb{R} \setminus Q$ in the system (10) then any initial point $(x, y) \in \mathbb{R}^2$ is non-wandering.

Theorem 3 (Peixoto 1961)**.** *Let f be a C^1-vector field on a compact two-dimensional differentiable manifold M. Then $\dot{x} = f(x)$ is structurally stable if and only if:*

(i) : *the number of critical points and cycles is finite and each is hyperbolic.*

(ii) : *there are no homoclinic or heteroclinic trajectories connecting saddles.*

(iii) : *the non-wandering set Ω consists of critical points and limit cycles only.*

This theorem provides an alternative check that systems in example 4.5 and theorem 2 are structurally unstable and stable, respectively.

4.3 Local bifurcations of continuous systems

In this paragraph, we consider continuous dynamical systems depending on parameter μ.

$$\dot{x} = f(x, \mu), \quad x \in \mathbb{R}^n,\ \mu \in\subset \mathbb{R}^p \tag{11}$$

Definition 4.5. (i) : $\mu = \mu_{cr}$ is called *a bifurcation value* for system (11) if the latter is not structurally stable at this parameter value.

(ii) : The *bifurcation set* is the set of locations in μ-space of bifurcation values.

(iii) : A *bifurcation diagram* is the set of locations in (x, μ) space of points with $f(x, \mu) = 0$ and μ belonging to the bifurcation set.

Necessary conditions for a bifurcation to occur at a point $(x_c,\ \mu)$ are given by:

$$\text{(i)} \quad f(x_c, \mu) = 0 \tag{12}$$

$$\text{(ii)} \quad 0 \in \operatorname{spec}\{D_x(x_c, \mu)\} \tag{13}$$

Indeed, condition (12) demands that x_c is a fixed point, while condition (13) that it is not hyperbolic (otherwise system (11) would be structurally stable by Theorem 2).

Here we will consider co-dimension 1 bifurcations, i.e. those whose bifurcation set

$$\Sigma = \{\mu \in \mathbb{R}^p : \exists x_c \in \mathbb{R}^n \text{ with } (x_c,\, \mu) \text{ satisfying conditions (12)-(13)}\}$$

has dimension $\dim \Sigma = p - 1$. For such bifurcations a generic line in μ-space crosses Σ at one point μ_c (see Fig. 4). Hence, there exists a system

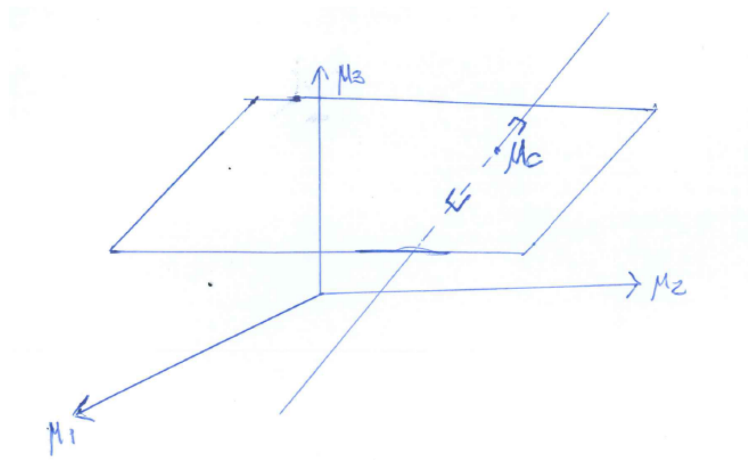

Figure 4: Example of a codimension-1 bifurcation set.

$$\dot{y} = g(y, \beta) \text{ with } y \in \mathbb{R}^n,\ \lambda \in \mathbb{R}, \tag{14}$$

which is topologically equivalent in the vicinity of the origin $(y,\, \beta) = (0,\, 0)$ to system (11) considered in the vicinity of $(x_c,\, \mu_c)$. System (14) is called the *normal form* for the bifurcation of (11) at $(x_c,\, \mu_c)$.

For the co-dimension 1 bifurcations condition (13) have necessarily one of the two following forms (see Fig. 5):

(i) *One simple zero eigenvalue:* In this case, the Jacobian at the bifurcation point

$$D_x f(x_c,\, \mu_c) = \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix},$$

where $A \in \mathrm{M}^{(n-1)\times(n-1)}$ is a matrix whose eigenvalues have non-zero real parts.

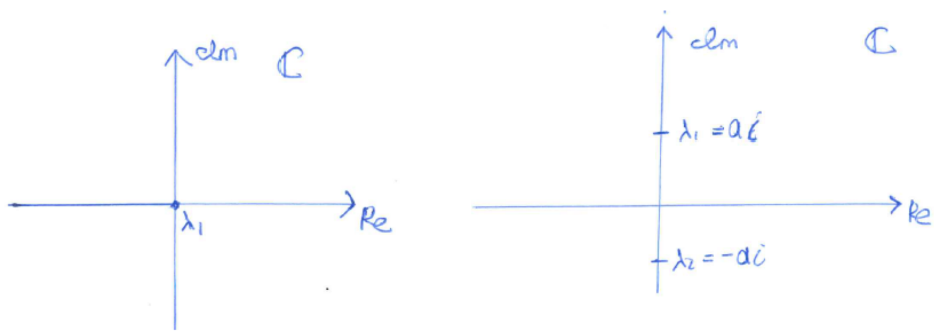

Figure 5: Two cases (i) and (ii) with eigenvalues crossing the imaginary axis for co-dimension 1 bifurcations.

(ii) *Two simple imaginary eigenvalues:* In this case the Jacobian at the bifurcation point

$$D_x f(x_c,\, \mu_c) = \begin{bmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & A \end{bmatrix},$$

where $A \in \mathrm{M}^{(n-2)\times(n-2)}$ is a matrix whose eigenvalues have non-zero real parts.

Let us first consider the case of simple zero eigenvalue in the one-dimensional case $n = 1$. The following theorem provides conditions for a so called *fold bifurcation* to occur at $(x,\, \mu) = (0,\, 0)$.

Theorem 4 (1D fold bifurcation)**.** *Suppose that a one-dimensional system*

$$\dot{x} = f(x,\, \mu),\; x \in \mathbb{R}^1,\; \mu \in \mathbb{R}^1, \tag{15}$$

with smooth f has at $\mu = 0$ the fixed point $x = 0$ and assume the following conditions:

$$\begin{aligned} \textit{(a)}\, f(0,0) &= 0 \;\textit{(fixed point)}, \\ \textit{(b)}\, f_x(0,0) &= 0 \;\textit{zero eigenvalue}, \\ \textit{(c)}\, f_{xx}(0,0) &\neq 0 \;\textit{non-degeneracy condition}, \\ \textit{(d)}\, f_\mu(0,0) &\neq 0. \end{aligned}$$

Then system (15) is topologically equivalent in the vicinity of the origin to the normal form system:

$$\dot{\eta} = \beta \pm \eta^2 + O(\eta^3) \;\textit{with}\; \eta \in \mathbb{R},\; |\beta| \ll 1, \tag{16}$$

where the signs + and − above correspond to the subcritical and supercritical cases, respectively.

Proof: consists of several steps.
Step 1: Expand $f(x, \mu)$ at (0, 0) in the Taylor expansion:

$$f(x, \mu) = f_0(\mu) + f_1(\mu)x + f_x(\mu)x^2 + O(x^3). \tag{17}$$

The conditions (a) and (b) above imply:

$$f_0(0) = f(0, 0) = 0 \quad \text{and} \quad f_1(0) = f_x(0, 0) = 0.$$

Let us introduce a new variable

$$\xi = x + \delta(\mu),$$

where $\delta(\mu)$ is unknown function. By substituting (4.3) into (17) and expanding in powers of ξ and δ one obtains:

$$\begin{aligned} \dot{\xi} &= \dot{x} = [f_0(\mu) - f_1(\mu)\delta + f_2(\mu)\delta^2 + O(\delta^3)] \\ &+ [f_1(\mu) - 2f_2(\mu)\delta + O(\delta^2)]\xi \\ &+ [f_2(\mu) + O(\delta)]\xi^2 + O(\xi^3). \end{aligned} \tag{18}$$

Step 2: By condition (c) one has $f_2(0) = 1/2 f_{xx}(0,0) \neq 0$. Hence, by the implicit function theorem for sufficiently small $\mu \ll 1$ there exists function $\delta(\mu)$ such that

$$\delta(\mu) = \frac{f_1(\mu)}{2f_2(\mu)} + O(\mu^2) \sim \frac{f_1'(0)}{2f_2(0)}\mu + O(\mu^2).$$

The last formula implies that the linear term in (18) is equal to zero. Therefore, with this function $\delta(\mu)$ one obtains:

$$\dot{\xi} = \gamma(\mu) + a(\mu)\xi^2 + O(\xi^3),$$

where

$$\begin{aligned} \gamma(\mu) &= f_0'(0)\mu + O(\mu^2), \\ a(\mu) &= f_2(0) + O(\mu). \end{aligned}$$

In turn, due to condition d) of the theorem one has:

$$\gamma(0) = 0 \text{ and } \gamma'(0) = f_0'(0) = f_\mu(0, 0) \neq 0.$$

Therefore, the inverse function theorem implies that for all sufficiently small $|\gamma| \leq 1$ there exists and inverse functions $\mu = \bar{\mu}(\gamma)$ and $\bar{a}(\gamma) = a(\bar{\mu}(\gamma))$ with $a(0) = f_2(0) \neq 0$ such that

$$\dot{\xi} = \gamma + \bar{a}(\gamma)\xi^2 + O(\xi^3) \tag{19}$$

Step 3: Let us finally introduce an invertible smooth change of variables in (19) $\eta = |\bar{a}(\gamma)|\xi$ and $\beta = |\bar{a}(\gamma)|\gamma$. Then (19) transforms into

$$\dot{\eta} = \beta \pm \eta^2 + O(\eta^3)$$

and the theorem is proved. □

Theorem 4 states that among the co-dimension 1 bifurcations the fold bifurcation is *generic*, meaning that if condition c) for the second derivative of the right-hand side of (15) holds then the normal form of the bifurcation is given by (16). From (16) one obtains that there are two fixed points with opposite signs for $\pm\beta > 0$ and none for $\pm\beta < 0$. One of the bifurcating fixed points is stable while another is unstable (see Fig. 6). In the cases when

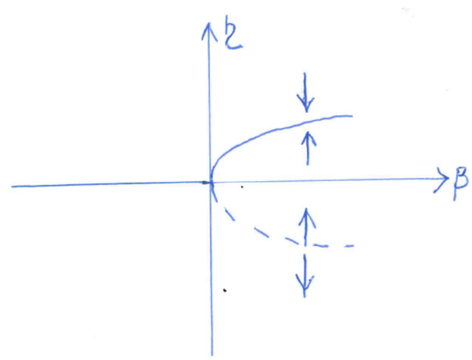

Figure 6: Bifurcation diagram for the supercritical fold normal form (16).

the *degeneracy condition* c) does not hold, the typical normal forms are (see Fig. 7)

$$\dot{y} = \beta y \pm y^2\text{-transcritical bifurcation} \tag{20}$$

or

$$\dot{y} = \beta y \pm y^3\text{-pitchfork bifurcation.} \tag{21}$$

Normal form (20) is generic for bifurcations in which $y = 0$ is an equilibrium for all $\beta \in \mathbb{R}$, while (21) is generic for bifurcations in which besides

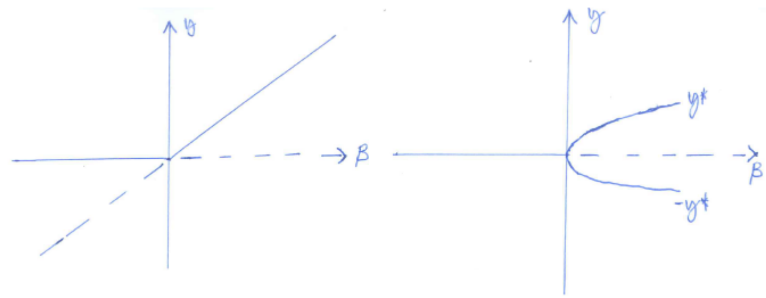

Figure 7: Bifurcation diagrams for the supercritical transcritical (20) (left) and pitchfork (21) (right) normal forms.

equilibrium $y = 0$ there are also two symmetric fixed points $\pm y^{\pm}$ having opposite signs.

Example 4.6 (non-generic normal form)**.** Let us consider system

$$\dot{y} = \beta - y^3. \tag{22}$$

It is degenerate because in contrast to (16) and (20)-(21) both derivatives $f_x(0,0) = f_{xx}(0,0) = 0$ and only $f_{xxx}(0,0) \neq 0$.

The corresponding curve of the fixed points (see Fig. 8(a)) given by $y(\beta) = \beta^{1/3}$ is locally asymptotically stable for all $\beta \in \mathbb{R}$, i.e. stability of system (22) does not change as β passes through zero. Hence $(y = 0,\, \beta = 0)$ is not a bifurcation point for (22). On the other hand, system (22) is not structurally stable: for any $\varepsilon > 0$ one can always find a perturbation $g_\varepsilon(y,\, \beta)$ of the right-hand side of (22) with

$$||g_\varepsilon(y,\, \beta)|| \leq \varepsilon, \tag{23}$$

such that (22) splits into two fold bifurcations at points $(\pm y_c, \pm\beta_c)$ with $y_c = O(\varepsilon)$ and $\beta_c = O(\varepsilon)$ (see Fig. 8(b)).

4.4 Bifurcations through reduction onto the extended center manifold

Let us now consider a general n-dimensional system

$$\dot{z} = F(z,\, \lambda) \text{ with } z \in \mathbb{R}^n,\ n > 1 \text{ and } \lambda \in \mathbb{R} \tag{24}$$

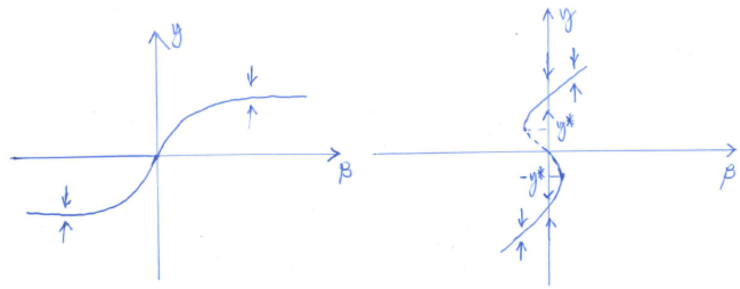

Figure 8: Curves of equilibria for system (22) (left) and its perturbation with (23).

having a bifurcation point at $(z_c,\ \lambda_c)$. Let us assume that Jacobian $D_zF(z_c,\ \lambda_c)$ has n distinct eigenvalues. We apply a change of variables:

$$z = z_c + C\tilde{z}, \quad \mu = \lambda - \lambda_c,$$

where matrix C diagonalise Jacobian:

$$C^{-1}D_zF(z_c,\ \lambda_c)C = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

with matrices A and B having dimensions equal to n_c (dimension of the central subspace E^c) and $n_c + n_u$ (sum of the dimensions of E^s and E^u). Then the system in variables $\tilde{z} = (x, y)^T$ and μ takes the form:

$$\begin{aligned} \dot{x} &= Ax + f(x, y, \mu), \\ \dot{y} &= By + g(x, y, \mu), \end{aligned} \tag{25}$$

where $x \in \mathbb{R}^{n_c}$ and $y \in \mathbb{R}^{n_s+n_u}$. We will consider possible bifurcations of system (25) at point $(x, y) = (0, 0)$ and $\mu = 0$. For this, we extend (25) by additional equation $\dot{\mu} = 0$ and consider

$$\dot{x} = Ax + f(x, y, \mu), \tag{26}$$
$$\dot{y} = By + g(x, y, \mu), \tag{27}$$
$$\dot{\mu} = 0, \tag{28}$$

the dynamical system in the extended phase-space $\mathbb{R}^m$ with $m = n_s + n_c + n_u + 1$. Let us construct the center manifold of system (26)-(28) at the origin, the coordinates along which are given by $(x,\, \mu)$. For that, we substitute ansatz $y = h(x, \mu)$ into equation (27) and solve it in the usual way to determine $h(x, \mu)$.

Once $h(x, \mu)$ is found we can write the dynamics in the extended center manifold $W^{cen}_{loc}(0, 0, 0)$ as

$$\dot{x} = A + f(x, h(x, \mu), \mu). \tag{29}$$

Equation (29) captures the relevant part of the system transformation occurring at the bifurcation. For the co-dimension 1 bifurcations with one simple zero, $n_c = 1$ and $x \in \mathbb{R}$, the normal form of equation (29) is usually given by (16) or (20)-(21).

Example 4.7. Consider system:

$$\begin{aligned} \dot{x} &= \mu(x + y) - (x + y)^2, & (30) \\ \dot{y} &= -y - \mu(x + y) + (x + y)^2, & (31) \\ \dot{\mu} &= 0. & (32) \end{aligned}$$

Its linearisation at the origin is given by:

$$D_x f(0, 0, \mu) = \begin{bmatrix} \mu & \mu \\ -\mu & -(\mu + 1) \end{bmatrix}$$

In the case $\mu = 0$ one has $\lambda_1 = 0$ and $\lambda_2 = -1$. Upon substitution of $y = h(x, \mu)$ into (31) one finds:

$$\partial_x h(x, \mu)\dot{x} = -y - \mu(x+y) + (x+y)^2 = -h(x, \mu) - \mu(x+h(x, \mu)) + (x+h(x, \mu))^2.$$

By substitution of $\dot{x}$ into the last equation from (30) one finds:

$$\partial_x h(x, \mu)[\mu(x+h(x, \mu)) - (x+h(x, \mu))^2] = -h(x, \mu) - \mu(x+h(x, \mu)) + (x+h(x, \mu))^2. \tag{33}$$

Let us now look for a quadratic approximation of *the extended local center manifold* at $(x, y, \mu) = (0, 0, 0)$:

$$h(x, \mu) = a_0\mu x + a_1\mu^2 + a_2x^2 + O(\mu^3 + x^3), \tag{34}$$

where the linear in μ and x terms are zero by the condition $W^{cen}_{loc}||E^c$. Substituting (34) into (33) and retaining only quadratic terms one obtains:

$$0 = -a_0\mu x - a_1\mu^2 - a_2x^2 - \mu x + x^2.$$

The last expression implies

$$a_0 = 0,\ a_1 = -1,\ a_2 = 1 \text{ and } h(x, \mu) = -\mu x + x^2.$$

Finally, substitution of found $h(x, \mu)$ into (30) gives the dynamics on the extended center manifold in the form:

$$\dot{x} = \mu(x + y) - (x + y)^2 = (1 - \mu)x(\mu - x),$$

which corresponds to the normal form of the transcritical bifurcation. □

Analogously to Theorem 4 formulated for the one-dimensional case, we would like sometimes to determine the type of bifurcations possible in the original system (24) by checking few conditions for its right-hand side $F(z,\ \lambda)$ at the bifurcation points rather then calculating the whole local center extended manifold at each point. The following theorem describes these conditions.

Theorem 5 (Sotomayor's theorem 1973)**.** *Let $A = DF(z = z_c, \lambda = \lambda_c)$ have one simple zero eigenvalue and vectors v and w be its right and left eigenvectors, respectively:*

$$Av = 0 \text{ and } w^T A = 0.$$

Define the following two numbers:

$$\alpha = \frac{1}{v \cdot w} w \cdot \frac{\partial F}{\partial \lambda}|_{(z=z_c,\, \lambda=\lambda_c)}, \tag{35}$$

$$\beta = \frac{1}{v \cdot w} \sum_{i,j,k=1}^{n} w_i v_j v_k \left(\frac{\partial^2 F_i}{\partial z_j \partial z_k} \right) |_{(z=z_c,\, \lambda=\lambda_c)}. \tag{36}$$

Part A: *If $\alpha \neq 0$ and $\beta \neq 0$ then there exists a smooth curve of the fixed points in $\mathbb{R}^n \times \mathbb{R}$ passing through $(z_c,\ \lambda_c)$ and tangent to $\mathbb{R}^n \times \{\lambda_c\}$ such that locally in the vicinity of $\lambda = \lambda_c$:*

(i) : either there are no fixed points for $\lambda < \lambda_c$ and two ones for $\lambda > \lambda_c$;

(i) : or there are no fixed points for $\lambda > \lambda_c$ and two ones for $\lambda < \lambda_c$;

Moreover, the dynamics in the corresponding extended center manifold parametrised by coordinates $(\eta,\ \lambda - \lambda_c)$ is given by the normal form of this fold bifurcation:

$$\dot{\eta} = \alpha(\lambda - \lambda_c) + \beta \eta^2$$

with α, β being defined above and $\eta = (z - z_c) \cdot v$ (projection at the one-dimensional center subspace).

Part B: *In the case $\alpha = 0$ define number*

$$\gamma = \frac{2}{v \cdot w} \sum_{i,j=1}^{n} w_i v_j \left(\frac{\partial^2 F_i}{\partial x_j \partial \mu} \right) |_{z=z_c,\, \lambda=\lambda_c} \tag{37}$$

In this case, the dynamics in the extended center manifold is one of the transcritical bifurcation:

$$\dot{\eta} = \gamma(\lambda - \lambda_c)\eta + \beta\eta^2, \tag{38}$$

with $\eta = (z - z_c) \cdot v$.

Example 4.8. Let us consider system

$$\begin{aligned} \dot{x} &= (1 + \lambda)x - 4y + x^2 - 2xy, \\ \dot{y} &= 2x - 4\lambda x - y^2 - x^2. \end{aligned} \tag{39}$$

At $(x, y, \lambda) = (0, 0, 1)$ Jacobian has a simple zero:

$$D = \begin{bmatrix} 2 & -4 \\ 2 & -4 \end{bmatrix}$$

with the corresponding right and left eigenvectors

$$v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad w = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and $v \cdot w = 1$. Calculating from formulae (35)-(36) and (37) one obtains:

$$\alpha = 0,\ \gamma = 12 \text{ and } \beta = 10.$$

Then according to (38) the bifurcation is transcritical with the dynamics in the extended center manifold:

$$\dot{\eta} = 12\eta(\lambda - 1) + 10\eta^2,$$

with $\eta = 2x + y$.

4.5 Hopf bifurcation

Let us consider now the co-dimension 1 bifurcations corresponding to the case *ii*) when two simple imaginary eigenvalues cross the imaginary axis, i.e. at the bifurcation point the Jacobian has the form:

$$D_x f(x_c, \mu_c) = \begin{bmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & A \end{bmatrix},$$

where $A \in \mathrm{M}^{(n-2)\times(n-2)}$ is a matrix whose eigenvalues have non-zero real parts. The generic bifurcation in this case is called Hopf bifurcation for which a periodic *limit circle* bifurcates from a fixed point.

We start consideration of the *supercritical Hopf bifurcation* by introducing its normal form:

$$\begin{aligned}\dot{x}_1 &= \mu x_1 - x_2 - x_1(x_1^2 + x_2^2),\\ \dot{x}_2 &= x_1 + \mu x_2 - x_1(x_1^2 + x_2^2).\end{aligned} \tag{40}$$

The system has one fixed point at the origin $(x_1, x_2) = (0, 0)$ for all $\mu \in \mathbb{R}$ with the Jacobian matrix

$$\begin{bmatrix} \mu & -1 \\ 1 & \mu \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

having eigenvalues $\lambda_\pm = \mu \pm i$. It is convenient to introduce the complex variable $z = x_1 + ix_2$ and rewrite system (40) in the following complex form:

$$\dot{z} = \dot{x}_1 + i\dot{x}_2 = \mu(x_1 + ix_2) + i(x_1 + ix_2) - (x_1 + ix_2)(x_1^2 + x_2^2),$$

i.e. to obtain the corresponding complex ODE:

$$\dot{z} = (\mu + i)z - z|z|^2. \tag{41}$$

Next, using the polar coordinate representation $z = \rho e^{i\varphi}$ with

$$\dot{z} = \dot{\rho} e^{i\varphi} + \rho i \dot{\varphi} e^{i\varphi}$$

one obtains from (41):

$$\dot{\rho} e^{i\varphi} + \rho i \dot{\varphi} e^{i\varphi} = \rho e^{i\varphi}(\mu + i - \rho^2),$$

which gives by equating the real and imaginary parts separately in the last equation the corresponding *polar form* of the system:

$$\dot{\rho} = \rho(\mu - \rho^2), \tag{42}$$

$$\dot{\varphi} = 1. \tag{43}$$

As equations (42) and (43) are uncoupled one can analyse the bifurcation of the phase portrait of the system as μ passes through zero using just ρ equation (42). From this it follows that the origin $\rho = 0$ is asymptotically stable for $\mu \leq 0$ and unstable for $\mu > 0$. Moreover, a new fixed point $\rho = \sqrt{\mu}$ of (42) bifurcates from the origin and exists for $\mu > 0$. It corresponds to the stable limit circle of the original system (40) (see Fig. 9). The next theorem provides conditions for which a general type two-dimensional system undergoes the Hopf bifurcation.

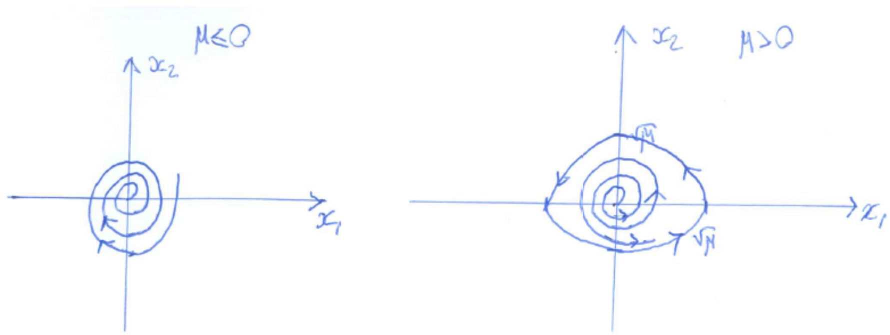

Figure 9: Supercritical Hopf bifurcation of the limit circle $\rho = \sqrt{\mu}$.

Theorem 6 (Hopf 1942)**.** *Consider the system*

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} \beta(\mu) & -\omega(\mu) \\ \omega(\mu) & \beta(\mu) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} f(u,v) \\ g(u,v) \end{bmatrix}, \tag{44}$$

with the nonlinear part being quadratic:

$$||f(u,v)|| = O(u^2+v^2), \quad ||g(u,v)|| = O(u^2+v^2).$$

Assume the following conditions hold:
$(a):$ $F(0,\,0,\,\mu) = 0$ *for* $\mu \in \mathbb{R}$,
$(b):$ $\mathrm{spec}\{D_x F(0,\,0,\,\mu)\} = \{\lambda_\pm\}$, *with*

$$\lambda_\pm(\mu) = \beta(\mu) \pm i\omega(\mu) \text{ and } \beta(0) = 0,\ \omega(0) = \omega_0 > 0, \tag{45}$$

$(c):$ *Lyapunov coefficient* $a \neq 0$, *where*

$$\begin{aligned} a &= \frac{1}{16\omega_0}\Big\{(f_{uuu} + f_{uvv} + g_{uuv} + g_{vvv})\omega_0 \\ &\quad + f_{uv}(f_{uu} + f_{vv}) - g_{uv}(g_{uu} + g_{vv}) - f_{uu}g_{uu} + f_{vv}g_{vv}\Big\}|_{(0,0)}, \end{aligned} \tag{46}$$

$(d):$ $d = \frac{d}{d\mu}\mathrm{Re}(\lambda(\mu))|_{\mu=0} \neq 0.$

Then there exists a smooth invertible change of variables such that system (44) can be transformed into the following form (considered in the polar coordinates):

$$\dot{\rho} = d\mu\rho + a\rho^3 + O(\rho^5), \tag{47}$$
$$\dot{\varphi} = \omega_0 + b\rho^2 + O(\rho^4). \tag{48}$$

Moreover, if $\operatorname{sign}(d/a) < 0$ *(*$\operatorname{sign}(d/a) > 0$*) then a limit circle with radius* $\rho(\mu) = \sqrt{-d\mu/a} + O(\mu)$ *bifurcates from the origin for* $\mu > 0$ *(*$\mu < 0$*). If* $d < 0$ *(*$d > 0$*) the origin is stable (unstable) for* $\mu > 0$*. The Hopf bifurcation is called supercritical (subcritical) if the bifurcating limit circle is stable (unstable).*

Remark 4.1. The minimal dimension in which Hopf bifurcation occurs is $n = 2$. Therefore, we consider two-dimensional system (9). Note that a general system of the form

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} f_1(x, y) \\ f_2(x, y) \end{bmatrix}$$

with $f_1, f_2 \in C^1(\mathbb{R}^2)$ which Jacobian has two simple conjugate eigenvalues (45) can be transformed into the form (9) via a linear change of variables

$$\begin{bmatrix} u \\ v \end{bmatrix} = C(\mu)^{-1} \begin{bmatrix} x \\ y \end{bmatrix},$$

where $C(\mu)$ is the matrix which columns are given by the eigenvectors of the Jacobian $v_\pm(\mu)$ corresponding to the eigenvectors $\lambda_\pm(\mu)$.

Remark 4.2. Conditions (c) and (d) of Theorem 6 are called *non-degeneracy* and *transversality* conditions, respectively. Indeed, the proof of the theorem shows that generically the term $a\rho^3$ is non-zero in (47), which corresponds to (c). Condition (d) insures that the pair of conjugate eigenvalues $\lambda_\pm(\mu)$ cross the imaginary axis as μ crosses zero.

Remark 4.3. By introducing additional change of variables and parameters one can show that system (47)-(48) is smoothly equivalent to (42)-(43) in the case $d/a < 0$ and $d > 0$. For given system (44) Theorem 6 provides a convenient way to calculate coefficients in system (47)-(48) using formulae for a and d in assumptions (c)-(d).

The next example shows how to apply Theorem 6 in order to calculate the polar normal form (47)-(48).

Example 4.9. Consider system

$$\begin{aligned} \dot{x} &= (1+\mu)x - 4y + x^2 - 2xy, \\ \dot{y} &= 2x - 4\mu y - y^2 - x^2. \end{aligned} \tag{49}$$

The two eigenvalues of system (49) are given by

$$\lambda_\pm = \frac{1}{2}\left(1 - 3\mu \pm \sqrt{-31 + 10\mu + 25\mu^2}\right).$$

At $\mu = 1/3$ the eigenvalues cross the imaginary axis, namely

$$\lambda_{\pm}(\mu = 1/3) = \pm\frac{2i\sqrt{14}}{3} \text{ and } v_{\pm} = (\frac{2}{3},\, 1) \pm i\left(\pm\frac{\sqrt{14}}{3},\, 0\right).$$

One can calculate

$$d = \mathrm{Re}\left(\partial_\mu \lambda|_{\mu=1/3}\right) = -3/2 \neq 0 \text{ and } \omega_0 = \frac{2\sqrt{14}}{3}.$$

Next, apply a linear change of variables to system (49) in order to bring it into the form (9). This change of variables is given by

$$\begin{bmatrix} u \\ v \end{bmatrix} = C^{-1}\begin{bmatrix} x \\ y \end{bmatrix},$$

where

$$C = \begin{bmatrix} 2 & -\sqrt{14} \\ 3 & 0 \end{bmatrix}.$$

System (49) in the variables $(u,\, v)$ takes the form:

$$\begin{aligned} \dot{u} &= -\frac{2\sqrt{14}}{3}v + f(u,v), \\ \dot{v} &= \frac{2\sqrt{14}}{3}u + g(u,v), \end{aligned}$$

with

$$\begin{aligned} f(u,\, v) &= \frac{1}{3}[-13u^2 + 4\sqrt{14}uv - 14v^2], \\ g(u,\, v) &= \frac{1}{21}[-\sqrt{14}u^2 + 14uv - 35\sqrt{14}v^2] \end{aligned}$$

Calculating now the Lyapunov coefficient a using formulae (50) one obtains

$$a = \frac{55}{84}.$$

We conclude that equation (47) takes the form:

$$\dot{\rho} = \left(-\frac{3}{2}(\mu - 1/3) + \frac{55}{84}\rho^2\right)\rho + O(\rho^5),$$

and, therefore, a *subcritical Hopf bifurcation* takes place at the origin at the parameter value $\mu = 1/3$ with a unstable limit circle bifurcating for $\mu > 1/3$.

4.6 Local bifurcations of 1D discrete maps

In this paragraph, we consider one-dimensional maps

$$x_{n+1} = f(x_n,\, \mu), \quad f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \tag{50}$$

depending on parameter μ. In this case, we define a fixed point x_c as a point which is mapped by f into itself:

$$f(x_c, \mu) = x_c. \tag{51}$$

Clearly, existence and form of x_c depends on the value of parameter μ. Let us also define a multiplier λ corresponding to the fixed point x_c as

$$\lambda = f_x(x_c,\, \mu_c). \tag{52}$$

We have shown in Part 3 of the lecture notes that if $|\lambda| < 1$ ($|\lambda| > 1$) then x_c is locally stable (unstable). The circle $|\lambda| = 1$ in the complex plane indicates change of stability and possibility for bifurcation of new fixed points from x_c. For λ lying on a unit circle below we will consider three different cases: $\lambda = 1$ or $\lambda = -1$ or $|\lambda| = 1$ with $\lambda \neq \bar{\lambda}$ (see Fig. 10). These three different

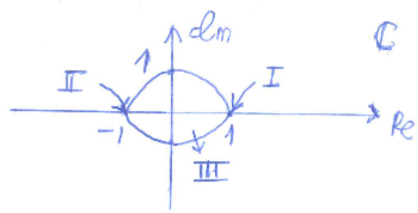

Figure 10: Bifurcation diagram for the *fold* normal form.

cases correspond to three different *codimension-1 bifurcations* of discrete maps. Note that the third case $|\lambda| = 1$ with $\lambda \neq \bar{\lambda}$ requires the dimension of the phase space to be at least $n = 2$. Therefore, for one-dimensional maps only codimension one bifurcations corresponding to the cases $\lambda = \pm 1$ are relevant. Below we consider these two cases separately.

4.6.1 Case I: $\lambda = 1$-fold bifurcation

The normal form of a fold bifurcation is given by

$$\eta_{n+1} = f(\eta_n, \beta) = \eta_n + \mu \pm \eta_n^2 + O(\eta_n^3) \tag{53}$$

The corresponding bifurcation diagram is presented in Fig. 11. For $\mu > 0$ ($\mu < 0$) there exist two fixed points $x_{\pm} = \pm\sqrt{\mu}$ ($x_{\pm} = \pm\sqrt{\pm\mu}$) of (53). One of the fixed points is stable and another one is unstable. Typical forms of function $f(x, \mu)$ for different parameter values μ are presented in Fig. 12

The next theorem (stated without proof) states that *fold bifurcation* is

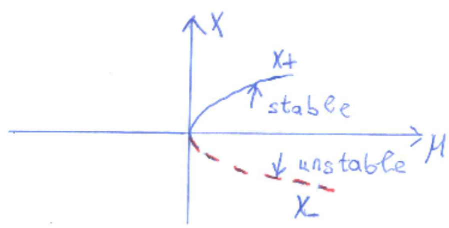

Figure 11: Bifurcation diagram for the *fold* normal form.

generic when $\lambda = 1$ holds in (52) at the bifurcation point $(x_c, \mu_c) = (0, 0)$.

Theorem 7. *For the map (50) let the following conditions be satisfied:*
(*a*) : *For* $\mu = 0$ *let* $x = 0$ *be a fixed point;*
(*b*) : $\lambda = f_x(0, 0) = 1$*;*
(*c*) : $f_{xx} \neq 0$ *(non-degeneracy condition);*
(*d*) : $f_\mu(0, 0) \neq 0$ *(transversality condition).*
Then system (50) is smoothly topologically equivalent to the normal form (53).

Similar to the continuous dynamical systems we state here also two other normal forms of (non-generic) bifurcations which are often met in applications:

$$\eta_{n+1} = \eta_n + \mu\eta_n - \eta_n^2 + O(\eta_n^3) \text{ transcritical }, \tag{54}$$
$$\eta_{n+1} = \eta_n + \mu\eta_n - \eta_n^3 + O(\eta_n^4) \text{ pitchfork }. \tag{55}$$

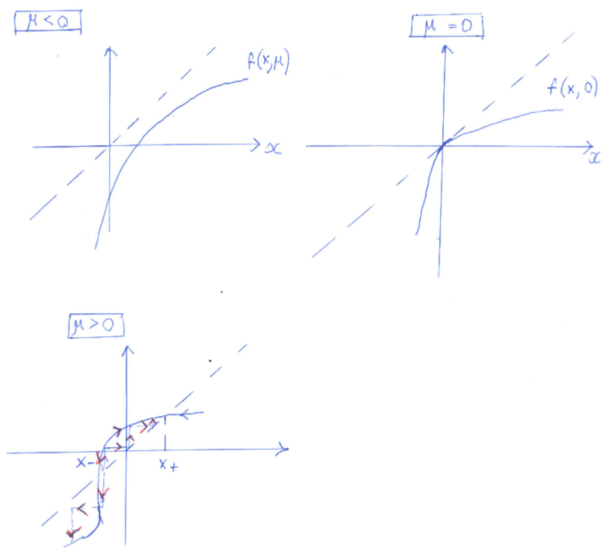

Figure 12: Typical forms of the map $f(x, \mu)$ exhibiting a fold bifurcation at (0, 0).

The connection between co-dimension one bifurcations of continuous dynamical systems ones for discrete maps is often provided by Poincare maps for periodic orbits of the former systems, which we introduced in Part 3 of these lecture notes.

Let us remind the definition of the Poincare maps. Consider a periodic (closed) solution curve Γ_μ of system

$$\dot{x} = f(x, \mu), \quad x \in \mathbb{R}^n, \, \mu \in \mathbb{R} \tag{56}$$

in the phase-space $\mathbb{R}^n$. For simplicity we assume that Γ_μ exists for all parameter values and smoothly depends on μ. Let us introduce a slice plane $\Sigma_\mu \subset \mathbb{R}^n$ which *transversely* intersects Γ_μ at a point x_μ and a neighborhood of this point $U_\mu(x_{mu}) \subset \mathbb{R}^n$. For each

$$x_1 \in U_\mu(x_\mu) \cap \Sigma_\mu \tag{57}$$

let us define a sequence of positive Poincare times t_1, t_2, ..., t_n, ... at which the flow started at x_1 hits subsequently the slice Σ again and again, i.e.

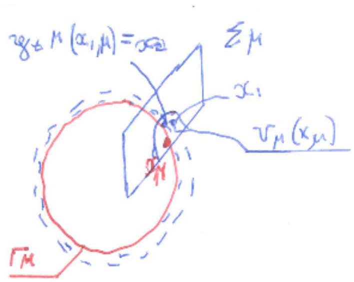

Figure 13: Illustration of periodic orbit Γ_μ and a Poincare section Σ_μ at its point x_μ.

there exists a sequence x_1, x_2, ..., x_n, ... with $x_n \in \Sigma_\mu$ defined recursively as

$$x_n = \varphi_t^\mu(x_{n-1}) \text{ for } n \in \mathbb{N}\,,$$

where φ_t^μ is the flow generated by (56) (see Fig. 13). Consequently, let us define $n-1$ dimensional Poincare map $P_\mu : U(x_\mu) \to V(x_\mu)$ corresponding to the periodic orbit Γ_μ of (56) via:

$$P_\mu(x_{n-1}) = x_n.$$

Naturally, $x_\mu \in \Gamma_\mu$ is the fixed point of P_μ and the question of asymptotic stability of the orbit Γ_μ is equivalent to asymptotic stability of x_μ for the map P_μ. The latter question can be analysed by calculating the multiplier $\lambda_\mu = P'_\mu(x_\mu)$. If $|\lambda_\mu| < 1$ ($|\lambda_\mu| > 1$) then the orbit Γ_{mu} is stable (unstable). At the parameter values μ_c for which $\lambda_{\mu_c} = 1$ new periodic orbits of system (56) may bifurcate from Γ_{μ_c}. The following example illustrates a fold bifurcation of a 1D Poincare map.

Example 4.10. Consider the system:

$$\begin{aligned} \dot{x} &= -y - x[\mu - ((x^2+y^2)^2 - 1)^2], \\ \dot{y} &= x - y[\mu - ((x^2+y^2)^2 - 1)^2]. \end{aligned}$$

It can be rewritten in the polar coordinates as

$$\dot{r} = -r[\mu - (r^2-1)^2], \tag{58}$$
$$\dot{\theta} = 1. \tag{59}$$

From (58) we obtain that the origin is the fixed point for all $\mu \in \mathbb{R}$ and for positive μ there are two periodic solutions with $r_\pm = \sqrt{1 \pm \mu^{1/2}}$. To analyse stability of the latter solutions let us introduce the Poincare section

$$\Sigma = \{(r, \theta) : \theta = \theta_b\}$$

for some $\theta_b \in (0, \pi/2)$ (see Fig. 14. From (59) one has that any point (r_1, θ_1)

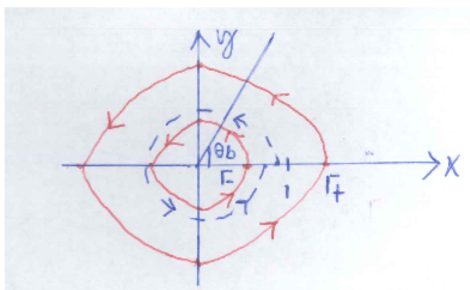

Figure 14: Periodic solutions $r = r_\pm$ and Poincare section Σ for system (58)–(59).

starting at Σ hits its again after Poincare time $T = 2\pi$. Denote the right-hand side of (58) by $f_\mu(r)$ and Taylor expand it in the vicinity of points $r_\pm$, respectively. Then (58) transforms into

$$\dot{r} = f'_\mu(r_\pm)(r - r_\pm) + O\left((r - r_\pm)^2\right).$$

By integrating the linear part of this equation one gets the leading order approximation of solutions:

$$r(t) \approx C \exp\{f'_\mu(r_\pm)t\} + r_\pm.$$

Correspondingly, define $\delta = r_1 - r_\pm$ and a linear map

$$P'_\mu(r_\pm) : \delta \to \delta \exp\{f'_\mu(r_\pm)2\pi\} \tag{60}$$

We observe that $\lambda_\pm(\mu) = P'_\mu(r_\pm)$ are multipliers of the Poincare maps $P_\mu(r_\pm)$ for periodic orbits $r = r_\pm$ of system (58)-(59). In our case,

$$f(r) = (\mu - (r^2 - 1))r, \quad f'(r_\pm) = 2(1 \pm \mu^{1/2}) \pm \mu^{1/2},$$

and hence

$$\lambda_\pm(\mu) = \exp\{\pm 8\pi\mu^{1/2}(1 \pm \mu^{1/2})\}.$$

One observes that for $0 < \mu \ll \infty$ one has $0 < r_- < 1 < r_+$ and, hence, $r = r_-$ is stable while $r = r_+$ is unstable (see Fig. 15). Moreover, $\mu_c = 0$ is the point of the *fold bifurcation* with $r_\pm = 1$ of the Poincare map $P_\mu(0)$, because $\lambda_\pm(0) = 1$ hold.

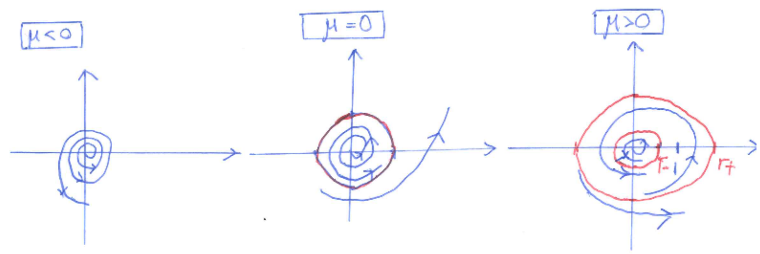

Figure 15: Phase plane portraits for system (58)–(59).

Alternatively, note that the same results presented in Fig. 15 can be obtained by analysing co-dimension one bifurcations of radial equation (58). The corresponding bifurcation diagram is presented in Fig. 16.

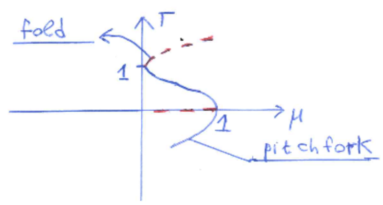

Figure 16: Bifurcation diagram for equation (58).

4.6.2 Case II: $\lambda = -1$-period doubling bifurcation

The normal form of the period doubling bifurcation is given by

$$x_{n+1} = f(x_n, \mu) = -x_n - \mu x_n \pm x_n^3 + O(x_n^4). \tag{61}$$

where + sign (− sign) corresponds to the supercritical (subcritical) case. One checks that

$$f(0,\,0) = 0 \text{ and } f_x(0,0) = \lambda = -1.$$

Let us consider the supercritical case in detail. At $\mu = -2$ the subcritical pitchfork bifurcation occurs when two solutions $x_\pm = \pm\sqrt{2+\mu}$ bifurcate from the origin. The origin itself is stable for $\mu \in (-2,\,0)$ and unstable otherwise. Let us calculate the second iterate of map (61):

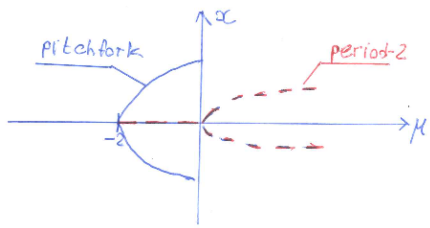

Figure 17: Bifurcation diagram for the normal form of the supercritical period doubling bifurcation.

$$f^2(x,\mu) = f(f(x,\mu),\mu) = x + \mu(2+\mu)x - 2x^3 + O(x^4). \tag{62}$$

The right-hand side of (62) turns out to be the normal form of the supercritical pitchfork bifurcation (compare with the normal form (55)) at $\mu = 0$. Also

$$f^2_x(0,\mu)|_{\mu=0} = \lambda = 1 + \mu(2+\mu)|_{\mu=0} = 1.$$

Correspondingly,

$$x_\pm = \pm\sqrt{\frac{\mu(2+\mu)}{2}} \tag{63}$$

are two fixed points of the second iterate f^2 of map (61). In turn, points (63) generate two stable period-2 orbits $\{x_1,\,x_2\}$ and $\{x_2,\,x_1\}$ of the original map (61). The full thus obtained bifurcation diagram for the normal form (61) (considered with + sign) is presented in Fig 17.

Period doubling bifurcation is very important type of map bifurcations, because it is often an elementary block for production of the so-called infinite countable *period doubling bifurcation cascades* which, in turn, lead to what is called *chaotic systems.* The next example gives a brief introduction to the famous *logistic map* discovered and analysed by Robert May and Mitchell

Feigenbaum in 1976-1977 that exhibits this complex type of the bifurcation behaviour.

Example 4.11. Let us consider the map (see Fig. 18)

$$x_{n+1} = \mu x_n(1 - x_n) \text{ for } 0 \le \mu \le 4 \text{ and } x_n \in (0,\, 1). \tag{64}$$

By applying standard analysis of this paragraph one obtains that at $\mu_1 = 1$

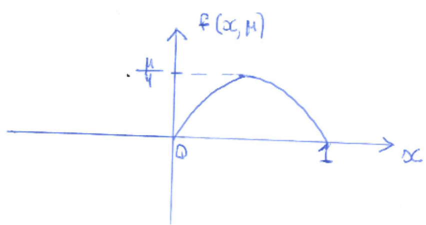

Figure 18: Plot of the logistic map $f(x,\, \mu) = \mu x_n(1 - x_n)$.

a transcritical bifurcation happens with solution $x_1 = 1 - 1/\mu$ bifurcating from $x_0 = 0$. Next, at $\mu_2 = 3$ a period doubling supercritical bifurcation occurs (see Fig. 19) with

$$x_{\pm,2} = \frac{\mu + 1 \pm \sqrt{(\mu - 3)(\mu + 1)}}{2\mu}.$$

In turn, at $\mu_3 = 1 + \sqrt{6}$ the second iterate f^2 of the map (64) experiences

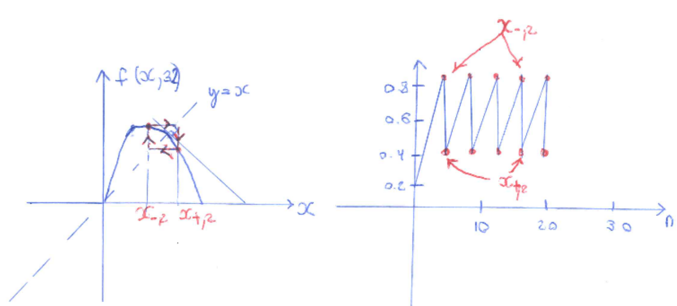

Figure 19: Plot of the logistic map $f(x, 3.2)$ (left) and the corresponding period-2 solutions (right).

another period doubling bifurcation at each of $x_{\pm,2}$ points that, correspondingly, produces four period-4 solutions of the original map (64) (see Fig. 20). Proceeding further one obtains a sequence of points $\{\mu_n\}$ such that for each

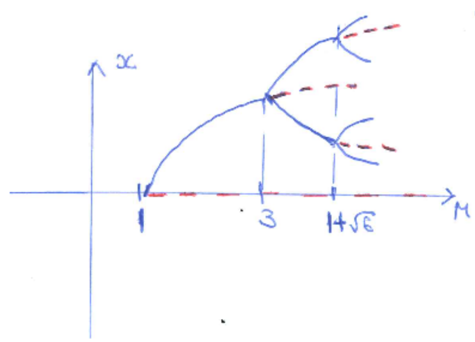

Figure 20: Plot of the first three bifurcation events of the logistic map.

$n \in \mathbb{N}$ the $2n$-th iterate of (64) f^{2n} experiences a period doubling bifurcation. Such behaviour is called as the period doubling cascade (see Fig. 21). Mitchell Feigenbaum observed three remarkable properties of it. Firstly, the sequence $\{\mu_n\}$ tends to some finite value

$$\lim_{n\to 0} \mu_n = \mu_\infty = 3.5699... \tag{65}$$

Secondly, the relative distance between the bifurcation points tends to another finite number:

$$\lim_{n\to\infty} \frac{\mu_k - \mu_{k-1}}{\mu_{k+1} - \mu_k} = \delta = 4.669201609... \tag{66}$$

Finally, the limiting numbers μ_∞ and δ are *universal*, in the sense that qualitatively the same period doubling cascade with another bifurcation sequence $\{\mu_n\}$ happens for any one-dimensional map $f(x,\mu)$ which is a convex function with a unique maximum in interval (0, 1) and attaining zero values at $x = 0, 1$ (i.e. having the shape similar to one in Fig. 18). Moreover, the limiting numbers μ_∞ and δ defined in (65)-(66), remarkably, are exactly the same for all $f(x,\mu)$ satisfying these simple conditions.

An additional observation can be made from the bifurcation diagram presented in Fig. 21. For $\mu > \mu_\infty$ there are no more periodic solutions and trajectories of the logistic map (64) become unstructured, see e.g. the right bottom plot in the figure. In this regime, the logistic map provides an interesting example of a chaotic dynamical system. In the next chapter

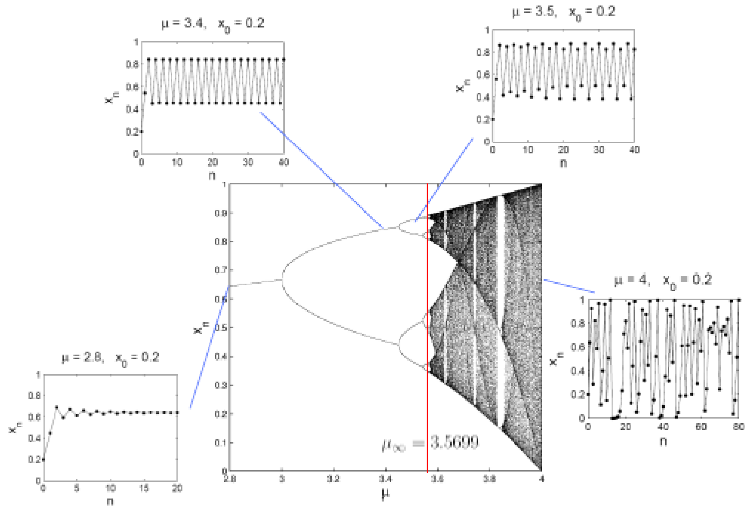

Figure 21: Plot of the full bifurcation diagram for the logistic map.

5 of these lecture notes we will consider few other discrete and continuous systems which exhibit chaotic behaviour and try to analyse the notion of chaos by means of these examples.