B5.6: Nonlinear Systems-Sheet 3 (solutions)

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Q1 (the complex Landau equation) The complex Landau equation

$$\dot{z} = az - b|z|^2 z,$$

for constants $a, b \in \mathrm{C}$.

We introduce the ansatz $z(t) = r(t)\exp(i\theta(t))$ for $r,\ \theta \in \mathbb{R}$ and split $a = a_1 + ia_2$ and $b = b_1 + ib_2$ to obtain from (1):

$$\dot{r} + ir\dot{\theta} = a_1 r + ia_2 r - b_1 r^3 - ib_2 r^3,$$

which can be separated into two real equations For the system

$$\begin{aligned} \dot{r} &= a_1 r - b_1 r^3, && (1)\\ \dot{\theta} &= a_2 - b_2 r^2. && (2) \end{aligned}$$

Given that $a_1 > 0$ from the question, we can rescale time $a_1 t \to t$ so that (1) depends on a single collection of parameters:

$$\dot{r} = r\left(1 - \frac{b_1}{a_1} r^2\right). \tag{3}$$

We note from (3) that equilibria radii occur at $r = \{0, \pm\sqrt{a_1/b_1}\}$, with the former being unstable and the latter two being asymptotically stable. On the other hand, (2) suggests that $\dot{\theta} = 0$ for radii $r = \pm\sqrt{a_2/b_2}$.

For periodic solutions to exist, we require that a_1 and b_1 are of the same sign; namely, that $b_1 > 0$, given that we have already assumed that $a_1 > 0$. For both $r > 0$ and $r < 0$, there is a single stable radius, however for periodic solutions to exist, we additionally must demand that $a_1/b_1 \neq a_2/b_2$. Without this constraint, we would have a fixed point rather than a periodic orbit. Lastly, we note that as $a_1/b_1 \to 0^+$, the three fixed radii will merge together leading to a supercritical pitchfork bifurcation.

Q2 (the glider) We study the equations of a simple glider:

$$\dot{y} = -\sin\theta - ay^2, \tag{4}$$

$$\dot{\theta} = y - \frac{\cos\theta}{y}. \tag{5}$$

For the case of zero drag $a = 0$, we differentiate the quantity $V = y^3 - 3y\cos\theta$ with respect to time and find:

$$\dot{V} = 3y^2\dot{y} - 3\dot{y}\cos\theta + 3y\dot{\theta}\sin\theta = 0,$$

meaning that V is conserved.

We note that the fixed points of (4)-(5) are $\sin\theta = n\pi$ for $n \in \mathrm{Z}\cap\{0\}$ and $y^2 = \cos\theta$. The latter solution suggests that there are only fixed points for even n and every fixed point is a center by linearization, whereby trajectories rotate anti-clockwise. The phase plot of (4)-(5) with $a = 0$, together with invariant sets determined by $V = y^3 - 3y\cos\theta$, is shown in Fig. 1. In the

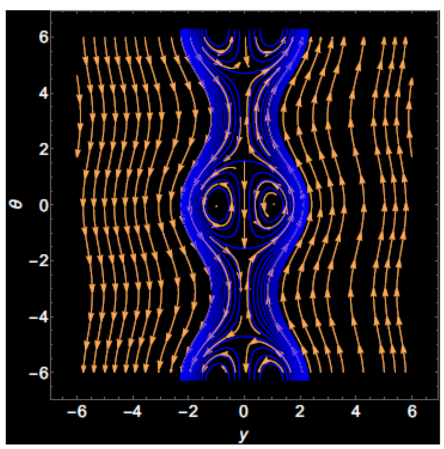

Figure 1: Phase plot of (4)-(5) with no drag $a = 0$ and invariant sets $V = y^3 - 3y\cos\theta$ for $V \in [-5, 5]$ in steps of 0.5 (shown in blue).

center, the glider bobs back and forth with periodic velocity that is always of a single sign, whilst out of the center, the glider does loop-de-loops with the same increasing and decreasing velocity that is always in a given direction.

With the inclusion of drag $a > 0$, the fixed points now occur for $\tan\theta = -a$ and $y^2 = \cos\theta$, which suggests $y = \pm 1/(1+a^2)^{1/4}$. We only consider the positive solutions in the forthcoming analysis, given that the negative solutions are dynamically the same by symmetry. The Jacobian in this case is given by:

$$Df = \begin{bmatrix} \frac{-2a}{(1+a^2)^{1/4}} & -\frac{1}{\sqrt{1+a^2}} \\ 2 & \frac{-a}{(1+a^2)^{1/4}} \end{bmatrix}$$

which has a trace, determinant, and discriminant given by:

$$\begin{aligned} \mathrm{tr}(Df) &= -\frac{3a}{(1+a^2)^{1/4}} < 0, \\ \det(Df) &= 2\sqrt{a^2+1} > 0, \\ \mathrm{tr}(Df)^2 - 4\det(Df) &= \frac{a^2-8}{\sqrt{1+a^2}}. \end{aligned}$$

Noting that $a > 0$, we use (8) to determine that the fixed points are stable spirals for $a < 2\sqrt{2}$ and stable nodes for $a > 2\sqrt{2}$. We show the phase plots of (4)-(5) in these respective regimes in Fig. 2.

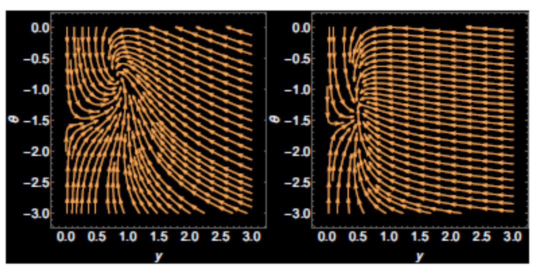

Figure 2: Phase plots of (4)-(5) for $a = 1$ (shown left) and $a = 3$ (shown right).

Q3 (non-wandering sets) It is important in the definition of the non-wandering property for a point p that arbitrarily large times t such that $\varphi_t(U) \cap U \neq \emptyset$ should exist for every possible neighborhood.

(i): Consider $\dot{\theta} = \mu - \sin\theta$ where $\theta \in S^1$. For $\mu > 1$, we note that there are no fixed points in the system with every point on the unit circle continuously proceeding anti-clockwise around the circle. As such, the non-wandering set includes all points on the unit circle. For $\mu = 1$, there is a fixed point at $\theta = \pi/2$ that is attracting in one direction, but repelling in the other. All trajectories proceed anti-clockwise around the circle. The smallest set that is non-wandering is the fixed point as any neighborhood that contains this fixed point has a non-zero intersection with any evolution of that neighborhood (i.e. the intersection being the fixed point itself). Lastly, for $0 \leq \mu < 1$, there are two fixed points, one of which is unstable and the other being asymptotically stable. Both fixed points are non-wandering and the non-wandering set is the union of both fixed points by similar reasoning.

(*ii*): Consider now $\ddot{\theta} + \sin\theta = 1/2$ where $\theta \in S^1$. We note that it is a conservative system which has a potential given by $V(\theta) = -\cos\theta - \theta/2$ so that the fixed points occur at $\theta = \{\pi/6, 5\pi/6\}$. The non-wandering set in this instance includes all the points that are enclosed by the homoclinic orbit beginning at $\theta = 5\pi/6$ and constrained by the potential $V \leq \sqrt{3}/2 - 5\pi/12$. Note that, even though $\theta \in S^1$, orbits that starts outside the homoclinic orbit are never periodic (i.e. their velocity increases after each turn). A phase plot showing the homoclinic orbit is shown in Fig. 3.

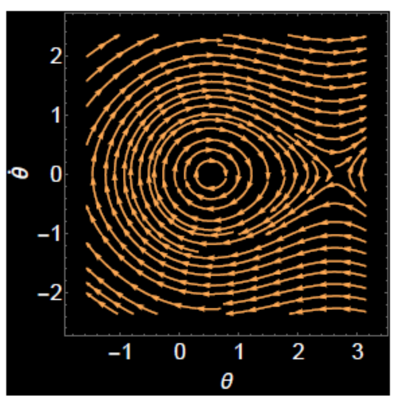

Figure 3: Phase plot of $\ddot{\theta} + \sin\theta = 1/2$ showing the homoclinic orbit.

Q4 (gradient vector fields) We study the gradient vector field:

$$\dot{x} = -\nabla V(x).$$

Proof for periodic solutions: In order to have a periodic that starts at t_0 and repeats after time t_1, we demand that:

$$V(x(t_1)) - V(x(t_0)) = 0. \tag{6}$$

However, we can rewrite the left hand side of (42) as:

$$\begin{aligned}
V(x(t_1)) - V(x(t_0)) &= \int_{t_0}^{t_1} \frac{dV}{dt}\,dt & (7)\\
\text{(chain rule)} &= \int_{t_0}^{t_1} \nabla V \cdot \dot{x}\,dt & (8)\\
\text{(using system (7))} &= \int_{t_0}^{t_1} -|\dot{x}|^2\,dt \leq 0, & (9)
\end{aligned}$$

with the equality only occurring if each point of the periodic orbit is a fixed point. As such, we have a contradiction with (6).

Proof for homoclinic solutions: By definition a homoclinic loop is an intersection of globally stable $W^s(x_0)$ and globally unstable $W^u(x_0)$ manifolds of a saddle fixed point. Therefore one has,

$$f(t) = V(x(t)) - V(x(-t)) \to 0 \quad \text{as} \quad t \to \infty. \tag{10}$$

On the other hand, similar to the argument used in (7)-(9) $f(t)$ is increasing function of t, because a homoclinic loop doesn't contain other fixed points. This fact contradicts to (10).

Q5 (the Lorenz system) We look at the classical Lorenz system:

$$\begin{aligned}
\dot{x} &= \sigma(y - x),\\
\dot{y} &= \rho x - xz - y,\\
\dot{z} &= xy - \beta z.
\end{aligned}$$

We consider a function having spheres as level sets $C(x, y, z) = x^2 + y^2 + (z - r - \sigma)^2$ and differentiate it with respect to t to obtain:

$$\begin{aligned}
\dot{C}(x, y, z) &= 2x\dot{x} + 2y\dot{y} + 2(z - r - \sigma)\dot{z}\\
&= 2x\sigma(y - x) + 2y(rx - xz - y) + 2(z - r - \sigma)(xy - \beta z)\\
&= -2\sigma x^2 - 2y^2 - 2\beta\left(z - \frac{r+\sigma}{2}\right)^2 + \frac{\beta(r+\sigma)^2}{2}. & (11)
\end{aligned}$$

Suppose the spherical surface given by $x^2 + y^2 + (z - \rho - \sigma)^2 = C$ has a big enough C so that it encloses the ellipsoid denoted by:

$$\frac{x^2}{\frac{\beta}{\sigma}\left(\frac{r+\sigma}{2}\right)^2} + \frac{y^2}{\beta\left(\frac{r+\sigma}{2}\right)^2} + \frac{\left(z - \frac{r+\sigma}{2}\right)^2}{\left(\frac{r+\sigma}{2}\right)^2} = 1, \tag{12}$$

which was obtained by considering $\dot{C} = 0$ from (11).

We note that trajectories starting on the surface with big enough C are outside of the ellipsoid given by (12), implying that $\dot{C} < 0$ from (11), so that the trajectories not only enter the sphere, but never escape it for all time. The spherical surface acts like a trapping region.