B5.6: Nonlinear Systems-Sheet 5

Dr. G. Kitavtsev

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Q1 (center manifold to fourth order) Consider the system

$$\dot{x} = xy + ax^3 + xy^2, \tag{1}$$
$$\dot{y} = -y + bx^2 + x^2y. \tag{2}$$

(i) : Show that the origin is asymptotically stable if $a + b < 0$ and unstable if $a + b > 0$.
(ii) : What happens is $a + b = 0$? Is the origin stable or unstable?

Q2 (topological equivalence) Prove that the system

$$\dot{x_1} = -x_1,$$
$$\dot{x_2} = -x_2,$$

is locally topologically equivalent near the origin to the system

$$\dot{x_1} = -x_1,$$
$$\dot{x_2} = -2x_2, \tag{3}$$

Are the systems smoothly and/or orbitally equivalent?

Q3 (extended centre manifold) Consider the system

$$\dot{x} = \mu x + y + \sin x, \tag{4}$$
$$\dot{y} = x - y. \tag{5}$$

Show that a bifurcation occurs at the origin of this system. Classify it and draw the local (close to the origin) phase portraits before and after the bifurcation..

Q4 (pitchfork bifurcation) Consider the equation $\dot{x} = f(x, \mu)$, where $x \in \mathbb{R}$ and f is at least of class C^3 in both arguments. Using the normal form stated in the lecture notes find the conditions on f such that a pitchfork bifurcation takes place at $x = \mu = 0$.

Q5 (Hopf bifurcation) The Brusselator is a simple model for a hypothetical chemical oscillator, named after the home of the scientists who proposed it. In dimensionless form, the dynamics is given by

$$\dot{x} = a - (b + 1)x + x^2y,$$
$$\dot{y} = bx - x^2y,$$

where a, $b > 0$ are parameters and x, $y \geq 0$ are dimensionless concentrations. Consider the case $a = 1$. Show that a Hopf bifurcation occurs at some parameter value $b = b_c$, where you should determine b_c. Find an approximation of the amplitude and the period of the limit cycle close to $b = b_c$.