B5.6: Nonlinear Systems-Sheet 6

Dr. G. Kitavtsev

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Q1 (Hopf bifurcation and bifurcation diagram) For equation

$$\ddot{x} + \mu\dot{x} + \nu x + x^2\dot{x} + x^3 = 0$$

identify types of bifurcations and the corresponding bifurcation curves in parameter plane (μ, ν). Conduct Hopf analysis at the parameter values where Hopf bifurcations occur. What happens at $\mu = \nu = 0$?

Q2* (Bifurcation diagram) Consider the system:

$$\begin{aligned}\dot{x} &= x - \sigma y - y(x^2 + y^2),\\ \dot{y} &= \sigma x + y - y(x^2 + y^2) - \gamma\end{aligned}$$

Using Sotomayor's and Hopf's theorems identify types of bifurcations and the corresponding bifurcation curves in parameter plane (σ, γ). (Note: the difficulty of this exercise is due to the analysis of a cubic equation for the fixed points. You should probably use symbolic computation to do the analysis).

Q3 (Binary expansion map) Consider the mapping $F : [0, 1] \to [0, 1]$:

$$F(x) = 2x \bmod 1.$$

(i) : Show that if $x \in [0, 1]$ has the binary expansion

$$x = .s_1 s_2 ... = \sum_{i=1}^{\infty} \frac{s_i}{2^i}$$

with $s_i \in \{0, 1\}$, then

$$F^n(x) = .s_{n+1} s_{n+2} ...$$

(ii) : Prove the existence of a countable infinity of periodic orbits.
(iii) : Prove the existence of an uncountable infinity of non-periodic orbits.
(iv) : Show that system has sensitive dependence to initial conditions.

Q4 (Existence of a homoclinic curve, Melnikov integral) Consider the perturbed Duffing equation

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= x - x^3 + \varepsilon[\alpha y + \beta x^2 y].\end{aligned}$$

Draw the phase portrait for $\varepsilon = 0$. Use Melnikov's method to find a condition on the parameters α and β such that two homoclinic orbits exist for ε small enough. Draw the perturbed phase portrait for $\varepsilon\alpha < 0$.

Q5 (Existence of transverse homoclinic points, Melnikov integral) Consider a gas bubble of volume $4\pi a^3/3$ in the presence of a time-periodic, axisymmetric uniaxial extensional flow of a fluid. The bubble shape is given by $r = r(\theta,\, t)$. Introduce a scalar measure of deformation:

$$x = \int_0^\pi r(\theta,\, t) P_2(\cos\theta)\sin\theta\, d\theta,$$

where P_2 is a Legendre polynomial. The variable x measures the deviation of the bubble from sphericity.

The evolution of x, in rescaled variables, is:

$$\ddot{x} = \omega - 2x + x^2 - \varepsilon(\mu\dot{x} + \delta\cos\omega t),$$

where ω is related to the Weber number:

$$W_0 = \frac{2\rho E_0^2 a^3}{\gamma}, \quad \omega = \frac{\mathrm{const}}{W_0 a^2},$$

where γ is the surface tension, E_0 the principal strain rate, ρ the fluid density and μ its viscosity.

After translation $x = -u + 1 + \sqrt{1-\omega}$, the system reads:

$$\ddot{u} + \omega_0^2 u + u^2 = \varepsilon(-\mu\dot{u} + \delta\cos(\omega t)),$$

where $\omega_0^2 = 2\sqrt{1-\omega}$.

For this system, compute the Melnikov function for the existence of transverse homoclinic points of the corresponding Poincare map. For a fixed forcing amplitude, find the optimal forcing frequency for bubble's breakup.