B5.6: Nonlinear Systems-Sheet 6 (solutions)

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Q1 (Hopf bifurcation and bifurcation diagram) We consider the equation

$$\ddot{x} + (\mu + x^2)\dot{x} + \nu x + x^3 = 0.$$

It can be split into two equations:

$$\begin{aligned} \dot{x} &= y, & (1) \\ \dot{y} &= -(\mu + x^2)y - \nu x - x^3, & (2) \end{aligned}$$

which has fixed points at $(x, y) = \{(0, 0), (\pm\sqrt{-\nu}, 0)\}$ and the corresponding eigenvalues:

$$\begin{aligned} \lambda|_{(0,0)} &= \frac{-\mu \pm \sqrt{\mu^2 - 4\nu}}{2}, & (3) \\ \lambda|_{(\pm\sqrt{-\nu},0)} &= \frac{-(\mu - \nu) \pm \sqrt{(\mu - \nu)^2 + 8\nu}}{2}. & (4) \end{aligned}$$

We observe that the case of a simple zero eigenvalue and interchange of stability happens for (3) and (4) when $\nu = 0$, $\mu \in \mathbb{R}$. In turn ,the case of two imaginary eigenvalues (corresponding to the Hopf bifurcation points) appears when $\mu = 0$ and $\nu > 0$ in (3) or $\mu = \nu$ and $\nu < 0$ in (4).

(i) : *Analysis of the Hopf bifurcations in the case* $\mu = 0$ *and* $\nu > 0$. In this case, system (1)-(2) takes the form

$$\begin{aligned} \dot{x} &= y, & (5) \\ \dot{y} &= -x^2 y - \nu x - x^3. & (6) \end{aligned}$$

The linearisation at the origin gives:

$$DF(0, 0) = \begin{bmatrix} 0 & 1 \\ -\nu & 0 \end{bmatrix} \qquad (7)$$

and two corresponding eigenvalues

$$\lambda_\pm = \pm i\sqrt{\nu},$$

with the corresponding eigenvectors

$$v_\pm = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mp i \begin{pmatrix} 1/\sqrt{\nu} \\ 0 \end{pmatrix}.$$

If we introduce the new variables $u = y$ and $v = \sqrt{\nu}x$ the system (5)-(6) transforms into the diagonal form:

$$\dot{v} = \sqrt{\nu}u, \tag{8}$$
$$\dot{u} = -\sqrt{\nu}v - \frac{1}{\nu}v^2 u - \frac{1}{\nu^{3/2}}v^3, \tag{9}$$

for which the Hopf theorem can be applied. In this case,

$$f(u,v) = -\frac{1}{\nu}v^2 u - \frac{1}{\nu^{3/2}}v^3, \quad g(u,v) = 0,$$

and the corresponding Lyapunov coefficient

$$\begin{aligned} a &= \frac{1}{16\sqrt{\nu}}\{(f_{uuu} + f_{uvv} + g_{uuv} + g_{vvv})\sqrt{\nu} + f_{uv}(f_{uu} + f_{vv}) \\ &\quad -g_{uv}(g_{uu} + g_{vv}) - f_{uu}g_{uu} + f_{vv}g_{vv}\}|_{u=v=0} = -\frac{1}{8\nu}. \end{aligned} \tag{10}$$

From (3) we can also calculate the coefficient d:

$$d = \mathrm{Re}\left(\frac{1}{2}\left(-1 \pm \frac{\mu}{\mu^2 - 4\nu}\right)|_{\mu=0}\right) = -\frac{1}{2}, \tag{11}$$

From (10)-(11) we obtain the normal form for this bifurcation:

$$\dot{r} = -\frac{\mu}{2}r - \frac{r^3}{8\nu},$$
$$\dot{\varphi} = \sqrt{\nu}.$$

We note from it that there are two fixed radii at $r = \{0, \sqrt{-4\mu\nu}\}$ for $\mu < 0$ and $r = 0$ for $\mu \geq 0$. The origin is asymptotically stable for $\mu \geq 0$, while it is unstable for $\mu < 0$ and the trajectories asymptotically tend to $r = \sqrt{-4\mu\nu}$, meaning the Hopf bifurcation is supercritical. Plots of (1)-(2) in these two respective regimes are given in Fig. 1.

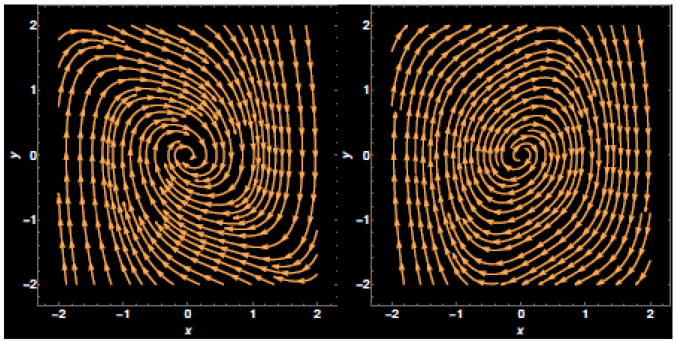

Figure 1: Plots of (1)-(2) for $\nu = 1$ and $\mu = 0.5$ (left) and $\mu = -0.5$ (right).

(ii) : Analysis of the Hopf bifurcations at the fixed points $(\pm\sqrt{-\nu}, 0)$ in the case $\mu = \nu$ and $\nu < 0$. In this case, system (1)-(2) can be written as

$$\dot{x} = y, \tag{12}$$
$$\dot{y} = -(\tilde{\mu} + \nu + x^2)y - \nu x - x^3, \tag{13}$$

where we introduced the shifted parameter $\tilde{\mu} = \mu - \nu$. Let us first analyse the bifurcation at $(\sqrt{-\nu}, 0)$. For $\tilde{\mu} = 0$ the eigenvalues $\lambda_\pm = \pm i\sqrt{-2\nu}$ are imaginary conjugate. We shift the fixed point to the origin via a linear change of variables:

$$\tilde{x} = x - \sqrt{-\nu}, \ \tilde{y} = y.$$

The system (12)-(13) transforms into

$$\dot{\tilde{x}} = \tilde{y}, \tag{14}$$
$$\dot{\tilde{y}} = 2\nu\tilde{x} - (\tilde{x}^2 + 2\tilde{x}\sqrt{-\nu})\tilde{y} - 3\tilde{x}^2\sqrt{-\nu} - \tilde{x}^3, \tag{15}$$

We observe that (14)-(15) is in almost diagonal form and introduce further linear change of variables:

$$u = \tilde{y}, \ v = \sqrt{-2\nu}\tilde{x}.$$

Accordingly, system (14)-(15) transforms into

$$\dot{v} = \sqrt{-2\nu}u, \tag{16}$$
$$\dot{u} = -\sqrt{-2\nu}v + \frac{1}{2\nu}v^2 u - \frac{3}{2\sqrt{-\nu}}v^2 + \frac{1}{2\nu\sqrt{-2\nu}}v^3 - \sqrt{2}vu, \tag{17}$$

for which the Hopf theorem can be applied. In this case,

$$f(u,v) = \frac{1}{2\nu}v^2 u - \frac{3}{2\sqrt{-\nu}}v^2 + \frac{1}{2\nu\sqrt{-2\nu}}v^3 - \sqrt{2}vu, \quad g(u,v) = 0,$$

and the corresponding Lyapunov coefficient

$$\begin{aligned} a &= \frac{1}{16\sqrt{-2\nu}}\{(f_{uuu} + f_{uvv} + g_{uuv} + g_{vvv})\sqrt{-2\nu} + f_{uv}(f_{uu} + f_{vv}) \\ & -g_{uv}(g_{uu} + g_{vv}) - f_{uu}g_{uu} + f_{vv}g_{vv}\}|_{u=v=0} = -\frac{1}{32\nu} > 0. \end{aligned} \tag{18}$$

Next, d coefficient can be calculated using (4) as

$$d = \mathrm{Re}\frac{d}{d\tilde{\mu}}\left[\frac{-\tilde{\mu} \pm \sqrt{\tilde{\mu}^2 + 8\nu}}{2}\right]|_{\tilde{\mu}=0} = -\frac{1}{2}. \tag{19}$$

Finally, combining (18) with (19) one obtains the normal form of the Hopf bifurcation for the case $\tilde{\mu} = \mu - \nu = 0$ and $\nu > 0$:

$$\begin{aligned} \dot{r} &= -\frac{1}{2}(\mu - \nu)r - \frac{1}{32\nu}r^3, \\ \dot{\varphi} &= \sqrt{-2\nu} \end{aligned}$$

In this case, the bifurcation is subcritical. For $\mu > \nu$ the fixed point $r = 0$ is stable, while the bifurcated limit circle $r = 4\sqrt{-(\mu - \nu)\nu}$ is unstable. For $\mu \leq \nu$ the unique fixed point $r = 0$ is unstable.

The analysis of the Hopf bifurcation for the fixed point $(-\sqrt{-\nu}, 0)$ proceeds similarly.

(iii) : Finally, for $\nu = \mu = 0$, there is a zero eigenvalue which is doubly degenerate. To find out the stability of the origin, we look for a Lyapunov function for it of the form:

$$V = cx^m + y^n,$$

where $c, m, n \in \mathbb{R}$ are constants to be determined. By differentiation one obtains

$$\dot{V} = mcx^{m-1}y - nx^2y^n - nx^3y^{n-1},$$

which suggests, by balancing the first and last term that $m = 4, n = 2$ and $c = 1/2$. In this case,

$$\dot{V} = -2x^2y^2 < 0 \;\forall\, (x, y) \setminus (0, 0),$$

implying that the origin is asymptotically stable. Furthermore, trajectories slowly spiral into the center in a clockwise manner, because at the points $(\pm\sqrt{\varepsilon}, 0)$ one has $\pm\dot{y} < 0$.

(iv) : We collect the results about the number, stability and type of the fixed points of system (1)-(2) at a bifurcation diagram in (ν, μ) plane presented in Fig. 2. We note that the curve $\nu = 0$, $\mu > 0$ ($\nu = 0$, $\mu < 0$) consists of supercritical (subcritical) Pitchfork bifurcation points; the curve $\mu = 0$, $\nu > 0$ contains supercritical Hopf bifurcation points for the fixed point $(0, 0)$; and the curve $\mu = \nu$, $\nu < 0$ contains subcritical Hopf bifurcation points for the fixed points $(\pm\sqrt{-\nu}, 0)$. The other two curves $\nu = \mu^2/4 > 0$ and $\mu = \nu \pm 4\sqrt{-2\nu}$, $\nu < 0$ contain the points where the fixed points change their type from being a node to a focus and vice versa.

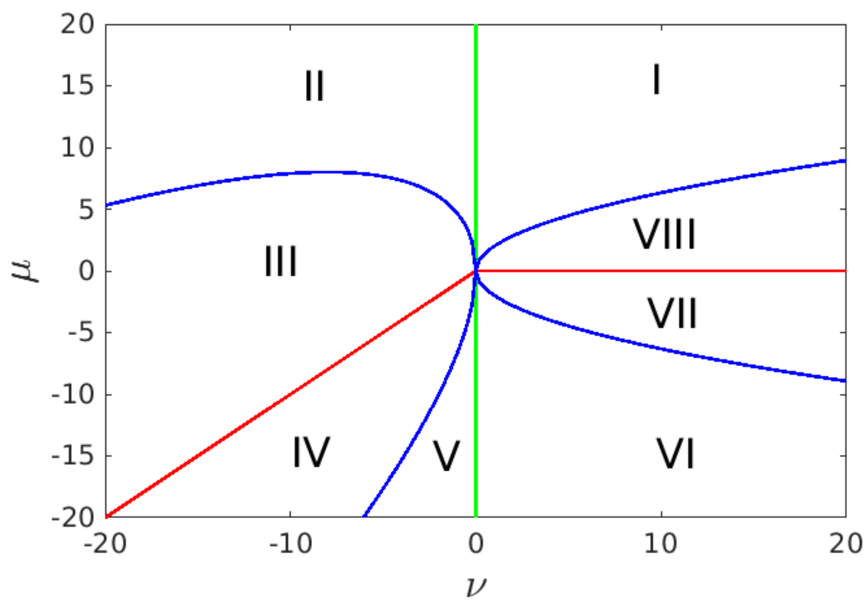

Figure 2: Bifurcation diagram for system (1)-(2) in (ν, μ) plane. Plotted are two Hopf bifurcation curves (in red), the Pitchfork bifurcation line (in green) and two curves where fixed points change their type (in blue). The curves split the plane into seven subdomains: I-the unique stable node $(0, 0)$; II-unstable $(0, 0)$ and two stable ($\pm\sqrt{-\nu}$, 0) nodes; III-unstable node $(0, 0)$ and two stable focuses ($\pm\sqrt{-\nu}$, 0), IV-unstable node $(0, 0)$ and two unstable focuses ($\pm\sqrt{-\nu}$, 0), V-three unstable nodes $(0, 0)$ and ($\pm\sqrt{-\nu}$, 0), VI-one unstable node $(0, 0)$, VII-one unstable focus $(0, 0)$, VIII-one stable focus $(0, 0)$.

Q3 (Binary expansion map) We consider the map $F : [0,1) \to [0,1)$:

$$F(x) = 2x \bmod 1, \tag{20}$$

and for each $x \in [0,1]$ the corresponding binary expansion

$$x = 0.s_1s_2... = \sum_{i=1}^{\infty} \frac{s_i}{2^i}. \tag{21}$$

We note that if $x < 1/2$, then $s_1 = 0$, whilst if $x \geq 1/2$, then $s_1 = 1$. Continuing to the second digit, if $x - s_1/2 < 1/4$, then $s_2 = 0$, whilst if $x - s_1/2 \geq 1/4$, one has $s_2 = 1$ and so forth.

(i) : We first prove that $F(x)$ shifts the binary expansion by one digit to the left and, if necessary, removes 1 from the first digit. Indeed,

$$F(x) = \sum_{i=1}^{\infty} \frac{s_i}{2^{i-1}} \bmod 1 = 0.s_2s_3...$$

Proving of $F^n(x) = 0.s_{n+1}s_{n+2}...$ for $n > 1$ proceeds by induction. Suppose that this is true for $n = m$,

$$F^m(x) = 0.s_{m+1}s_{m+2}s_{m+3}...,$$

then

$$F^{m+1}(x) = \sum_{i=1}^{\infty} \frac{s_{m+i}}{2^{i-1}} \bmod 1 = 0.s_{m+2}s_{m+3}...,$$

as required.

(ii) : We note that the countably infinite number of rational numbers within the region $[0,1)$ have binary expansions which are n-periodic for $n \in \mathbb{N}$. As such, applying (20) n times leads to the same rational number, thus resulting in an n-periodic orbit. Given that there are countably infinite rational numbers in $[0,1]$, this implies that (20) allows a countably infinite number of orbits.

(iii) : We observe that there are an uncountably infinite number of irrational numbers within the region $[0,1)$ which, by definition, have binary expansions that are non-periodic. Applying (20) to such x any number of times will never result in the same original number, given this non-periodicity. As such, we conclude that (20) allows for an uncountably infinite number of non-periodic orbits.

(iv) : F has sensitive dependence on initial conditions if $\exists \delta > 0$, such that for any $x \in [0,1)$ and $\varepsilon > 0$ there exist $y \in [0,1)$ and $n > 0$ such that $|x-y| <$

ε and $|F^n(x) - F^n(y)| > \delta$. To prove this property, we choose $1/2^{n+1} < \varepsilon$, so that, if we have some number x expressed as a binary expansion, then by changing its $(n+1)$-th binary digit, we obtain y. Mathematically, that is:

$$|x - y| = \frac{1}{2^{n+1}} < \varepsilon,$$

by definition (21).

Applying (20) n times, we find:

$$|F^n(x) - F^n(y)| = \frac{1}{2},$$

given that x and y differ in their $(n+1)$-th binary digit. Choosing $\delta = 1/3$, we note that:

$$|F^n(x) - F^n(y)| > \delta,$$

implying that the map F has sensitive dependence on initial conditions for all $x \in [0, 1)$.

Q4 (Existence of a homoclinic curve, Melnikov's integral) We consider system

$$\dot{x} = y, \tag{22}$$
$$\dot{y} = x - x^3 + \varepsilon(\alpha y + \beta x^2 y), \tag{23}$$

For $\varepsilon = 0$, we find the first integral of the unperturbed Hamiltonian system in the form:

$$\frac{1}{2}\dot{x}^2 + V(x) = E = \text{const}, \tag{24}$$

with potential $V(x) = \frac{x^4}{4} - \frac{x^2}{2}$. By plotting potential V we determine that there are three fixed points $(\pm 1,\, 0)$ and $(0,\, 0)$ to the unperturbed system with the former two being centers and the latter a saddle (see Fig. 3). For the saddle point and two homoclinic curves starting and ending at $(0, 0)$, one calculates $E = V(0) = 0$. Using this, we integrate equation (24):

$$t - t_0 = \int \frac{dx}{x\sqrt{1 - \frac{x^2}{2}}}, \tag{25}$$

to obtain the explicit form of two homoclinic curves $\pm(\bar{x}(t), \dot{\bar{x}}(t))$.

Using the substitution

$$s = \sqrt{1 - x^2/2},\; dx = -\frac{2s}{x} ds,$$

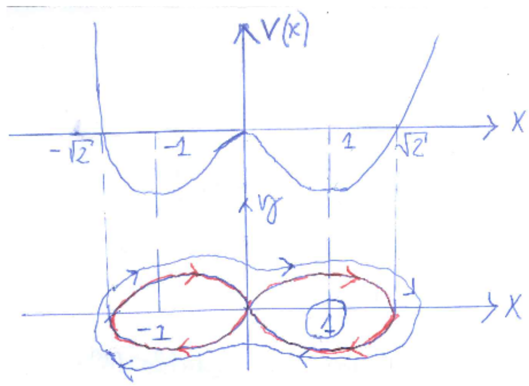

Figure 3: Sketch of the potential $V(x)$ and the phase plane for system (22)-(23) with $\varepsilon = 0$.

one calculates

$$\int \frac{dx}{x\sqrt{1-\frac{x^2}{2}}} = -\frac{1}{2}\log\left[\frac{1+s}{1-s}\right]_{s=\sqrt{1-x^2/2}}. \tag{26}$$

Assuming $\bar{x}(0) = \sqrt{2}$, i.e. $t_0 = 0$, one deduces from (25) and (26) that

$$(\bar{x}(t), \dot{\bar{x}}(t)) = (\sqrt{2}\text{sech}(t), -\sqrt{2}\text{sech}(t)\tanh(t)). \tag{27}$$

Next, we consider the perturbed system (22)-(23) with $|\varepsilon| \ll 1$ and calculate the corresponding Melnikov's integral:

$$M(\alpha, \beta) = \int_{-\infty}^{+\infty} f(\bar{x}_0(t)) \wedge g(\bar{x}_0(t), \alpha, \beta)\, dt,$$

with

$$f(x, y) = \begin{pmatrix} y \\ x - x^3 \end{pmatrix} \quad \text{and} \quad g(x, y) = \begin{pmatrix} 0 \\ \alpha y + \beta x^2 y \end{pmatrix}.$$

Namely,

$$\begin{aligned} M(\alpha, \beta) &= \int_{-\infty}^{\infty} [\alpha y^2 + \beta x^2 y]\, dt \\ &= \int_{-\infty}^{\infty} [2\alpha \text{sech}^2(t)\tanh^2(t) + 4\beta \sec^4(t)\tanh^2(t)]\, dt \\ &= \frac{4}{15}(5\alpha + 4\beta), \end{aligned} \tag{28}$$

where we used the formulae:

$$
\begin{aligned}
\int_{-\infty}^{\infty} \tanh^k(t)\mathrm{sech}^2(t)\, dt &= \frac{2}{k+1} \text{ for } k \in \mathbb{N} \text{ being even,} \\
\mathrm{sech}^2(t) &= 1 - \tanh^2(t).
\end{aligned}
$$

To determine when the two homoclinic orbits persist to exist in system (22)-(23) for all sufficiently small ε, we look for simple zeros of (28) and find that $\alpha = -4\beta/5$ is the necessary constraint to ensure existence.

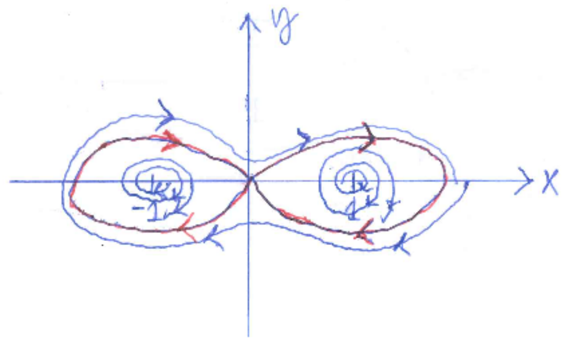

Figure 4: A sketch of the phase plane for system (22)-(23) with $\varepsilon\alpha < 0$ and $\alpha = -4\beta/5$.

For $\varepsilon\alpha < 0$ and $\alpha = -4\beta/5$ the phase portrait is sketched in Fig. 4. The set of fixed points is independent of ε. Points $(\pm 1, 0)$ become unstable focuses while $(0, 0)$ remains being the saddle. The union of the perturbed homoclinic curves form an attracting set: the trajectories inside and outside it asymptotically tend to it.

Q5 (Existence of transverse homoclinic points, Melnikov's integral) We consider the system

$$
\begin{aligned}
\dot{u} &= v, &(29)\\
\dot{v} &= -\omega_0^2 u - u^2 + \varepsilon(-\mu v + \delta\cos(\omega t)). &(30)
\end{aligned}
$$

In the case $\varepsilon = 0$, the unperturbed Hamiltonian system admits the first integral:

$$
\frac{1}{2}\dot{u}^2 + V(u) = E = \text{const}, \tag{31}
$$

with potential function

$$V(u) = \frac{u^3}{3} + \omega_0^2 \frac{u^2}{2}.$$

By determining zeros and extremal points of V one deduces that there exists a homoclinic curve $(\bar{u}_0(t), \dot{\bar{u}}_0(t))$ connecting the saddle $(-\omega_0, 0)$ with itself (see Fig. 5). For the saddle point and the homoclinic curve

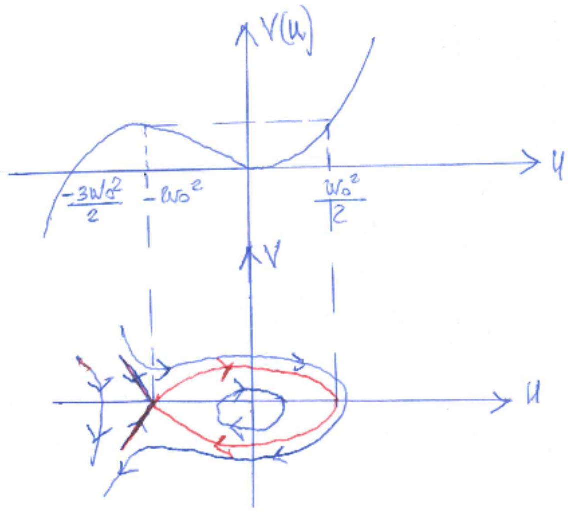

Figure 5: Sketch of the potential $V(u)$ and the phase plane for system (29)-(30) with $\varepsilon = 0$.

$$E = V(-\omega_0^2) = \frac{\omega_0^6}{6}.$$

To determine the second root of the level set $V(u) = E$ we write

$$V(u) - E = \frac{u^3}{3} + \omega_0^2 \frac{u^2}{2} - \frac{\omega_0^6}{6} = c(u + w_0^2)^2(u - b). \tag{32}$$

By comparison of the coefficients in the last polynomial one finds $c = 1/3$ and the second root $b = \omega_0^2/2$. Now, using (32) one can integrate the equation (31) for the homoclinic curve $(\bar{u}_0(t), \dot{\bar{u}}_0(t))$:

$$\dot{u} = \pm \frac{1}{\sqrt{3}} \sqrt{(u + \omega_0^2)^2(\omega_0^2 - 2u)}$$

as

$$\frac{1}{\sqrt{3}}(t - t_0) = \int \frac{du}{(u + \omega_0^2)\sqrt{\omega_0^2 - 2u}}. \tag{33}$$

Employing the change of variables:

$$s = \sqrt{\omega_0^2 - 2u}, \quad du = -s ds,$$

one obtains

$$\int \frac{du}{(u+\omega_0^2)\sqrt{\omega_0^2 - 2u}} = -\frac{1}{\sqrt{3}\omega_0} \log \left[\frac{s + \sqrt{3}\omega_0}{\sqrt{3}\omega_0 - s} \right] |_{s=\sqrt{\omega_0^2 - 2u}}$$

Substituting this into (33) and choosing $\bar{u}_0(0) = \omega_0^2/2$, i.e. $t_0 = 0$, one obtains the explicit solution:

$$(\bar{u}_0(t), \dot{\bar{u}}_0(t)) = \left(\frac{3}{2}\omega_0^2 \text{sech}^2(\omega_0 t/2) - \omega_0^2,\ -\frac{3}{2}\omega_0^3 \text{sech}^2(\omega_0 t/2) \tanh(\omega_0 t/2) \right). \tag{34}$$

Next, we calculate the Melnikov's integral

$$M(t_0) = \int_{-\infty}^{+\infty} f(\bar{u}_0(t)) \wedge g(\bar{u}_0(t), t + t_0)\, dt,$$

with

$$f(u,v) = \left(\begin{array}{c} v \\ -\omega_0 u - u^2 \end{array} \right) \quad \text{and} \quad g(u,v,t) = \left(\begin{array}{c} 0 \\ -\mu v + \delta \cos(\omega t) \end{array} \right).$$

Using (34) one obtains

$$\begin{aligned} M(t_0) &= M_1 + M_2(t_0) = -\frac{9\mu\omega_0^6}{4} \int_{-\infty}^{\infty} \tanh^2(\omega_0 t/2) \text{sech}^4(\omega_0 t/2)\, dt \\ &\quad -\frac{3\delta\omega_0^3}{2} \int_{-\infty}^{\infty} \tanh(\omega_0 t/2) \text{sech}^2(\omega_0 t/2) \cos(\omega(t+t_0))\, dt. \end{aligned} \tag{35}$$

Let us calculate the integrals M_1 and M_2 separately. Using the formulae

$$\begin{aligned} \int_{-\infty}^{\infty} \tanh^k(\tau)\text{sech}^2(\tau)\, dt &= \frac{2}{k+1} \text{ for } k \in \mathbb{N} \text{ being even,} \\ \text{sech}^2(\tau) &= 1 - \tanh^2(\tau), \end{aligned}$$

and the change of variables $\tau = \omega_0 t/2$ one calculates

$$M_1 = -\frac{6\mu\omega_0^5}{5}. \tag{36}$$

For calculation of M_2 we again use the change of variables $\tau = \omega_0 t/2$ to write it as

$$\begin{aligned} M_2(t_0) &= -3\delta\omega_0^2 \int_{-\infty}^{\infty} \tanh(\tau)\mathrm{sech}^2\tau \cos(2\omega\tau/\omega_0 + \omega t_0)\, d\tau \\ &= 3\delta\omega_0^2 \sin(\omega t_0) \int_{-\infty}^{\infty} \tanh(\tau)\mathrm{sech}^2(\tau) \sin(2\omega\tau/\omega_0)\, d\tau, \end{aligned} \quad (37)$$

where in the last line we used that $\tanh(\tau)\mathrm{sech}^2(\tau)$ is an odd function. Next, using the method of calculating residue in the extended complex plane (see Perko book, problem 4, pages 430-431), one can calculate:

$$\int_{-\infty}^{\infty} \tanh(\tau)\mathrm{sech}^2(\tau) \sin(\alpha\tau)\, d\tau = \frac{\pi\alpha^2}{2\sinh(\pi\alpha/2))}.$$

Using $\alpha = 2\omega\tau/\omega_0$ in the last formula and (37), one calculates:

$$M_2(t_0) = \frac{6\delta\omega^2\pi \sin(\omega t_0)}{\sinh(\omega\pi/\omega_0)}.$$

Combining the last formula with (35) and (36) one obtains the expression for Melnikov's integral:

$$M(t_0) = -\frac{6\mu\omega_0^5}{5} + \frac{6\delta\omega^2\pi \sin(\omega t_0)}{\sinh(\omega\pi/\omega_0)}.$$

Setting $M(t_0) = 0$ is equivalent to

$$\sin(\omega t_0) = \frac{\mu\omega_0^5 \sinh(\omega\pi/\omega_0)}{5\delta\omega^2\pi}.$$

Therefore, a necessary condition for a chaotic behaviour of a bubble is given by the one for which there is a simple zero of the last expression, i.e.

$$\left|\frac{\mu\omega_0^5 \sinh(\omega\pi/\omega_0)}{5\delta\omega^2\pi}\right| < 1$$

holds. For given fixed μ and δ the optimal frequency ω can be find from the last inequality.