Question Sheet 0

These questions are intended largely for revision, and won't necessarily be covered in the class, though please ask about them if you're unsure. (i) is recommended especially for students who have not done B5.3 Viscous Flow, (ii) is a reminder of the wave equation, and (iii) reiterates derivations covered in the first two lectures.

1. Let $V(t)$ be a time-dependent closed region of $\mathbb{R}^3$ that is convected with velocity $\boldsymbol{u}(\boldsymbol{x},t)$. Prove *Reynolds' Transport Theorem*, namely

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V(t)} f(\boldsymbol{x},t)\,\mathrm{d}V = \iiint_{V(t)} \frac{\partial f}{\partial t} + \boldsymbol{\nabla}\cdot(f\boldsymbol{u})\,\mathrm{d}V,$$

for any continuously differentiable function $f(\boldsymbol{x},t)$.

2. (a) Consider the first order wave equation for $u(x,t)$ on $-\infty < x < \infty$, $t > 0$:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0, \qquad u(x,0) = f(x).$$

By performing a change of variables $(x,t) \mapsto (\eta,t)$, where $\eta = x - ct$, show that the solution is $u = f(x - ct)$. Making your own choice for $f(x)$, sketch the solution as a function of x at $t = 0$ and $t = 1$. Derive the same result using the Method of Characteristics.

(b) Recall the wave equation

$$\frac{\partial^2 \phi}{\partial t^2} = c^2\frac{\partial^2 \phi}{\partial x^2}.$$

By performing a change of variables $(x,t) \mapsto (\xi,\eta)$ to suitably chosen characteristic variables, show that the general solution is $\phi(x,t) = F(x+ct) + G(x-ct)$, where F and G are arbitrary functions.

3. Let ρ, $\boldsymbol{u}$, p and T denote the density, velocity, pressure and temperature of an inviscid fluid, and let $\boldsymbol{g}$ be the acceleration due to gravity. You may assume that the pressure, density and temperature are related through the ideal gas law $p = \rho RT$.

Starting from conservation of mass, momentum and energy for a material volume, use the Transport Theorem to derive the equations

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{u}) = 0, \qquad \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\frac{1}{\rho}\boldsymbol{\nabla}p + \boldsymbol{g}, \qquad \rho c_v\frac{\mathrm{D}T}{\mathrm{D}t} = -p\boldsymbol{\nabla}\cdot\boldsymbol{u} + \boldsymbol{\nabla}\cdot(k\boldsymbol{\nabla}T).$$

[*You may assume that the thermal energy is c_vT per unit mass, where the specific heat c_v is constant, and use Fourier's law of conduction, with thermal conductivity k.*]

Define the entropy S per unit mass in terms of the other variables, and show that

$$\rho T\frac{\mathrm{D}S}{\mathrm{D}t} = \boldsymbol{\nabla}\cdot(k\boldsymbol{\nabla}T).$$

Deduce that, if gas is confined in a stationary thermally-insulated container D, then the rate of change of total entropy is

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_D \rho S\,\mathrm{d}V = \iiint_D \frac{k|\boldsymbol{\nabla}T|^2}{T^2}\,\mathrm{d}V \geqslant 0.$$