Question Sheet 2

1. Consider an ocean of depth h between a fixed base $z = -h$ and an upper surface $z = 0$ held at constant atmospheric pressure p_a. The fluid in the ocean is incompressible and starts in a stratified static state with density $\rho = \rho_0 F(z)$. Show that the corresponding pressure is given by

$$p_0(z) = p_a + \rho_0 g \int_z^0 F(\zeta)\,\mathrm{d}\zeta.$$

The fluid is perturbed by small-amplitude two-dimensional disturbances, such that the density, pressure and velocity fields are given by

$$\rho = \rho_0 F(z) + \rho'(x,z,t), \qquad p = p_0(z) + p'(x,z,t), \qquad \boldsymbol{u} = u(x,z,t)\hat{\boldsymbol{e}}_x + w(x,z,t)\hat{\boldsymbol{e}}_z,$$

and the free surface lies at $z = \eta(x,t)$ (but remains at pressure p_a). Show that w satisfies the equation

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\right) = \frac{1}{F}\frac{\mathrm{d}F}{\mathrm{d}z}\left(g\frac{\partial^2 w}{\partial x^2} - \frac{\partial^3 w}{\partial z \partial t^2}\right).$$

and write down the boundary conditions that apply at the bottom $z = -h$ and on the free surface $z = \eta$. Linearise the free surface conditions to $z = 0$, and write them as a single condition on w. [*Considerable care is required with this linearisation.*]

If waves propagate such that $w = \mathrm{Re}\left\{A(z)\mathrm{e}^{\mathrm{i}(kx-\omega t)}\right\}$, derive an equation for $A(z)$ and show that the appropriate boundary conditions are

$$A(-h) = 0\,, \qquad \omega^2\frac{\mathrm{d}A}{\mathrm{d}z}(0) - k^2 g A(0) = 0\,.$$

For high-frequency waves, the condition at $z = 0$ can be approximated by $\mathrm{d}A/\mathrm{d}z = 0$. In this case, and supposing that $F(z) = \mathrm{e}^{-\lambda z}$, show that the frequency of such waves must satisfy

$$\omega^2 < \frac{4\lambda g k^2}{\lambda^2 + 4k^2}.$$

2. A barotropic compressible fluid, for which the pressure p and density ρ are related by $p = P(\rho)$, is contained in a rectangular pipe between rigid walls at $x = 0$, $x = a$, $y = 0$, $y = b$, a rigid end at $z = 0$ and a free end at $z = h$ maintained at constant pressure p_0.

Assuming that gravity is negligible, show that small-amplitude sound waves may be described using a velocity potential ϕ that satisfies the three-dimensional wave equation with speed c_0, which you should define. Also find the boundary conditions for ϕ on each of the boundaries.

Find the natural frequencies of oscillation and show that, if a and b are both very much smaller than h, then the lowest frequencies are approximately of the form

$$\omega = \left(j + \frac{1}{2}\right)\frac{\pi c_0}{h},$$

for non-negative integer j.

Compare with the equivalent result when the end at $z = 0$ is also maintained at pressure p_0, and hence describe the difference in pitch (the fundamental frequency) between 'stopped' and 'open' organ pipes of the same length.

3. Small-amplitude waves disturb a gas contained in a two-dimensional box $0 < x < L$, $0 < y < b$. The flow is described by a velocity potential ϕ that satisfies the wave equation with speed c_0.

   (a) If one side of the box is driven, so that

   $$\frac{\partial \phi}{\partial x} = a\mathrm{e}^{-\mathrm{i}\omega t} \quad \text{(real part assumed) at } x = 0,$$

   write down appropriate boundary conditions for the other three sides, and obtain the solution

   $$\phi = \frac{ac_0 \cos\big(\omega(L-x)/c_0\big)}{\omega \sin(\omega L/c_0)} \mathrm{e}^{-\mathrm{i}\omega t}.$$

   For which values of ω is this solution invalid? What happens if ω takes one of these values?

   (b) Now suppose $L = \infty$, and

   $$\frac{\partial \phi}{\partial x} = a \cos\left(\frac{\pi y}{b}\right) \mathrm{e}^{-\mathrm{i}\omega t} \quad \text{at } x = 0.$$

   Show that

   $$\phi = \cos\left(\frac{\pi y}{b}\right) \mathrm{e}^{-\mathrm{i}\omega t} \left(A\mathrm{e}^{\lambda x} + B\mathrm{e}^{-\lambda x}\right)$$

   is a possible solution provided

   $$\lambda^2 = \frac{\pi^2}{b^2} - \frac{\omega^2}{c_0^2}.$$

   Explain how to determine the constants A and B, distinguishing carefully between the cases $0 < \omega < \pi c_0/b$ and $\omega > \pi c_0/b$. [*Hint: think about what condition must be applied at infinity.*]

4. Acoustic waves are generated in a compressible fluid by small oscillations of a cylinder, whose radius at time t is given in plane polar coordinates by

   $$r = a\left(1 + \epsilon \mathrm{e}^{-\mathrm{i}\omega t}\right) \quad \text{(real part assumed).}$$

   The flow is described by a velocity potential ϕ satisfying the two-dimensional wave equation, with sound speed c_0. Assuming ϵ is small, derive the approximate boundary condition

   $$\frac{\partial \phi}{\partial r} = -\epsilon\omega a\mathrm{i}\mathrm{e}^{-\mathrm{i}\omega t} \quad \text{at } r = a.$$

   Show that separable radially-symmetric solutions are given by

   $$\phi(r,t) = \mathrm{e}^{-\mathrm{i}\omega t}\left\{A\mathrm{J}_0\left(\frac{\omega r}{c_0}\right) + B\mathrm{Y}_0\left(\frac{\omega r}{c_0}\right)\right\},$$

   where A and B are constants, and $\mathrm{J}_0(\xi)$ and $\mathrm{Y}_0(\xi)$ are Bessel functions.

   Given that $\mathrm{J}_0(\xi) \to 1$ and $\mathrm{Y}_0(\xi) \to -\infty$ as $\xi \to 0$, and that $\mathrm{J}_0(\xi)$ and $\mathrm{Y}_0(\xi)$ have the asymptotic behaviour

   $$\mathrm{J}_0(\xi) \sim \sqrt{\frac{2}{\pi\xi}} \cos(\xi - \pi/4), \qquad \mathrm{Y}_0(\xi) \sim \sqrt{\frac{2}{\pi\xi}} \sin(\xi - \pi/4),$$

   as $\xi \to \infty$, determine the constants A and B for (a) the waves inside the cylinder, and (b) the waves outside the cylinder. [*Hint: think about what condition must be applied at infinity.*]