B4.2 Functional Analysis II - Sheet 1 of 4

Read Sections 1.1-1.3 and prove the few statements whose proofs were left out as an exercise. (Not to be handed in.)

Do:

Q1. Let $(X, \|\cdot\|)$ be a real norm vector space satisfying the parallelogram law:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \text{ for all } x, y \in X.$$

Define

$$f(x,y) = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) \text{ for } x, y \in X.$$

Show that

(a) $f(x,y) = f(y,x)$.

(b) $f(x+z,y) = f(x,y) + f(z,y)$.

(c) $f(\alpha x, y) = \alpha f(x,y)$ for all $\alpha \in \mathbb{R}$.

Conclude that $f(x,y)$ defines an inner product on X.

[*Hint: The tricky part is (c). Prove it, successively, for α being an integer, a rational number, and finally a real number.*]

Q2. Let $A^2(\mathbb{D})$ be the Bergman space of functions which are holomorphic and square integrable on the unit disk $\mathbb{D} \subset \mathbb{C}$. Let $f \in A^2(\mathbb{D})$, $0 < s < 1$, and $|z| < s$. Cauchy's integral formula gives

$$r\, f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta}) r d\theta$$

for any $0 < r < 1 - s$.

(a) Integrating the above formula in $0 < r < 1 - s$, show that

$$f(z) = \frac{1}{\pi(1-s)^2} \langle f, \chi_{D(z,1-s)} \rangle_{L^2(\mathbb{D})},$$

where $D(z, 1-s)$ is the disk of radius $1 - s$ with the centre z.

(b) Deduce that

$$|f(z)| \leq \frac{\|f\|_{L^2(\mathbb{D})}}{\sqrt{\pi}(1-s)}.$$

(c) Deduce that if f_n is a Cauchy sequence in $A^2(\mathbb{D})$ then f_n converges uniformly on compact subsets of $\mathbb{D}$.

(d) Deduce that $A^2(\mathbb{D})$ is closed in $L^2(\mathbb{D})$.

Q3. Let K be a non-empty convex set of a real Hilbert space X. Suppose that $x \in X$ and $y \in K$. Prove that the following are equivalent:

(1) $\|x-y\| \leq \|x-z\|$ for all $z \in K$;

(2) $\langle x-y, z-y\rangle \leq 0$ for all $z \in K$.

Q4. Let Y be a subspace of a Hilbert space X over $\mathbb{C}$ and $\ell : Y \to \mathbb{C}$ be a bounded linear functional on Y.

(a) Using the Riesz representation theorem, show that there is a unique extension of ℓ to a bounded linear functional $\tilde{\ell}$ on X with $\|\tilde{\ell}\|_{X^*} = \|\ell\|_{Y^*}$.

(b) By examining the behavior of $\tilde{\ell}$ on the orthogonal complement of Y, reprove (a) without using the Riesz representation theorem.

Q5. For each of the cases below, determine – in any order – (i) the orthogonal complement of Y in X, (ii) if Y is dense in X, and (iii) if Y is closed in X. Here all spaces are over the real.

(i) $X = L^2(-1,1)$, $Y = \{f \in X : \int_{-1}^{1} f(x)\,dx = 0\}$.

(ii) $X = \ell^2$, $Y = \{(a_n) \in X : a_2 = a_4 = \ldots = 0\}$.

(iii) $X = L^2(0,1)$, $Y = C[0,1]$.

In (a) and (b) you may find it useful to rewrite the identities defining the space Y as an orthogonal relation e.g. $a_2 = 0$ means $\langle a, e_2\rangle = 0$.

Q6. Let Y be the set of all $g \in L^2(-\pi,\pi)$ such that $g(t-\pi) = g(t)$ for almost all $t \in (0,\pi)$. Show that Y is a closed subspace of $L^2(-\pi,\pi)$ and identify $Y^\perp$. Assume that $f \in L^2(-\pi,\pi)$ and supposed $f = g + g^\perp$, where $g \in Y$ and $g^\perp \in Y^\perp$. Find g and $g^\perp$.

Calculate

$$d := \inf\{\|h - g\|_{L^2(-\pi,\pi)} : \ g \in Y\},$$

where $h(t) = t$ and specify the element g at which the infimum is attained.